PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 2, February 1979

A NOTE ON GENERATING RELATIONS FOR LAURICELLA'S FUNCTION OF SEVERAL VARIABLES

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ABSTRACT. In this paper we have derived generating relations for Lauricella's function by using the operator $T_k = x(k + xD)$ and the operational relations involving this operator. Some recent results of Srivastava [3], Srivastava and Carlitz [4], Sharma and Abiodun [1] have been conveniently obtained by this method as well as some hitherto unknown results established.

1. Introduction. Srivastava [2], [3] considered the function

$$f(z_1, \ldots, z_r) = \sum_{k_1, \ldots, k_r = 0}^{\infty} C_{k_1, \ldots, k_r} \prod_{j=1}^{r} \frac{(z_j)^{k_j}}{k_j!}, \tag{1.1}$$

where $C_{k_1, \ldots, k_r}$ are arbitrary complex constants, and derived the generating relation (2.2). Considering the function in (1.1), Carlitz and Srivastava [4] obtained several generating functions for Lauricella's function of several variables.

Agarwal [8] and Sharma and Abiodun [1] considered the general G-function (4.1) and Sharma and Abiodun [1] derived a generating relation for it.

In this paper, making use of operational methods, we propose to derive results of Srivastava [3], Carlitz and Srivastava [4], Sharma and Abiodun [1] and other results directly by making use of the operator formulas (2.1), (3.1), (3.4), (3.6) and (3.8).

In §2 we obtain a generating relation for $f(z_1, \ldots, z_r)$ defined in (1.1) and, in §3, we derive various generating functions for Lauricella's function of several variables. In §4 we consider the general G-function and, using (2.1), we derive directly the result due to Sharma.

The series occurring in the paper may be thought of as formal power series.

2. Generating relations for functions of several variables. In what follows we shall make use of the following Mittal [7] operational generating formula

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} T^n_{a+1+(m-1)n} f(x) = \frac{(1+v)^{a+1}}{1-(m-1)v} f[x(1+v)] \tag{2.1}$$

Received by the editors March 31, 1977.
AMS (MOS) subject classifications (1970). Primary 33A65.
Key words and phrases. Lauricella's functions of several variables, generating relations.

where $v = xz(1+v)^m$, m being constant and $f(x)$ admits a formal power series in x and $T_k \equiv x(k + xD)$, $D = d/dx$.

Assuming that the operator T_k operates on t alone and putting $a = \alpha$, $m = \beta + 1$ and

$$f(x) \equiv f\left[x_1(-y^{-\beta}t^{1+\beta})^{m_1}, \ldots, x_r(y^{1-\beta}t^{\beta})^{m_r}\right] \quad \text{in (2.1)}$$

we immediately get

$$\sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n} t^n z^n \sum_{k_1, \ldots, k_r = 0}^{n} \frac{(-n)_{\Sigma m_i k_i} z^{-\Sigma m_i k_i}}{(1+\alpha+\beta n)_{\Sigma m_i k_i}}$$
$$\cdot C(k_1, \ldots, k_r) t^{\beta \Sigma m_i k_i} y^{-\beta \Sigma m_i k_i} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!}$$
$$= (1+v)^{\alpha+1}(1-\beta v)^{-1} f\left[x_1\left[-y^{-m_1\beta}\left\{t^{\beta+1}(1+v)^{\beta+1}\right\}^{m_1}\right], \ldots, x_r\left[-y^{-m_r\beta}\left\{t^{\beta+1}(1+v)^{\beta+1}\right\}^{m_r}\right]\right], \tag{2.2}$$

where $v = tz(1+v)^{\beta+1}$. Now putting $z = 1$ and $y = t$ in (2.2) we get

$$\sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n} t^n \sum_{k_1, \ldots, k_r = 0}^{n} \frac{(-n)_{\Sigma m_i k_i}}{(1+\alpha+\beta n)_{\Sigma m_i k_i}}$$
$$\cdot C(k_1, \ldots, k_r) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!}$$
$$= (1+v)^{\alpha+1}(1-\beta v)^{-1} f\left[x_1(-v)^{m_1}, \ldots, x_r(-v)^{m_r}\right] \tag{2.3}$$

where $v = t(1+v)^{\beta+1}$, which is due to Srivastava [2].

3. Lauricella's function of several variables. In this section we shall use the Mittal [6] operational formulae

$$\sum_{n=0}^{\infty} \frac{1}{n!} T_{a+1}^n \{f(x)\} = (1-x)^{-a-1} f\left[\frac{x}{1-x}\right] \tag{3.1}$$

where $f(x)$ admits a formal power series in x. Putting $a = \lambda - 1$, and

$$f(x) \equiv F_D^r\left[(\lambda, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i); x_1(-t)^{m_1}, \ldots, x_r(-t)^{m_r}\right]$$

in (3.1) and assuming that T_k operates on t alone, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_D^{(r)}\left[(-n, m_i): (\beta_i, \varphi_i); (\gamma_i, \psi_i); x_1, \ldots, x_r\right] t^n$$
$$= (1-t)^{-\lambda} F_D^{(r)}\left[(\lambda, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i); x_1\left(\frac{t}{t-1}\right)^{m_1}, \ldots, x_r\left(\frac{t}{t-1}\right)^{m_r}\right], \tag{3.2}$$

which is due to Carlitz and Srivastava [4]. Carlitz and Srvistava [4] also proved that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n(\gamma)_n t^n}{n!\,(\mu)_n} F_D^{(r)}[(-n, m_i): (\beta_i, \varphi_i); (\gamma, m_i); x_1, \ldots, x_r]$$

$$= (1-t)^{-\lambda} F_D^{(r+1)}\Bigg[(\lambda, m_i, 1): (\beta_i, \varphi_i), (\mu-\gamma, 1); (\mu; m_i, 1);$$

$$x_1\left(\frac{t}{t-1}\right)^{m_1}, \ldots, x_r\left(\frac{t}{t-1}\right)^{m_r}, \frac{t}{t-1}\Bigg]. \quad (3.3)$$

Putting $a = \lambda - 1$ and

$$f(x) = F_D^{(r+1)}[(\lambda, m_i, 1): (\beta_i, \varphi_i), (\mu-\gamma, 1); (\mu, m_i, 1);$$

$$x_1(-t)^{m_1}, \ldots, x_r(-t)^{m_r}, (-t)]$$

in (3.1) and assuming that T_k-operates on t alone, and using Euler's transformation, the result in (3.3) follows. Again by making use of the Mittal [6] operational formula

$$\sum_{n=0}^{\infty} \frac{1}{n!} T_{a+n}^{n}\{f(x)\} = (1-4x)^{-1/2}\left[\frac{2}{1+(1-4x)^{1/2}}\right]^{a-1}$$

$$\cdot f\left[\frac{2x}{1+(1-4x)^{1/2}}\right]. \quad (3.4)$$

Putting $a = \lambda$ and

$$f(x) = F_D^{(r)}[(\lambda, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i); x_1 t^{m_1}, \ldots, x_r t^{m_r}]$$

in (3.4) and assuming that T_k-operates on t alone, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_{2n} t^n}{n!\,(\lambda)_n} F^{2:1,\ldots,1}_{2:0,0,\ldots,0}\left[\begin{matrix}(\lambda, m_i), (-n, m_i): (\beta_i, \varphi_i); x_1, \ldots, x_r \\ (\gamma, \psi_i), (1-\lambda-2n, m_i)\end{matrix}\right]$$

$$= (1-4t)^{-1/2}\left[\frac{2}{1+(1-4t)^{1/2}}\right]^{\lambda-1}$$

$$\cdot F_D^{(r)}\Bigg[(\lambda, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i);$$

$$x_1\left(\frac{2t}{1+(1-4t)^{1/2}}\right)^{m_1}, \ldots, x_r\left(\frac{2t}{1+(1-4t)^{1/2}}\right)^{m_r}\Bigg]. \quad (3.5)$$

Putting $\lambda = \gamma$ and $\psi_i = m_i$ in (3.5) we get the following generating relation for

the Lauricella function

$$\sum_{n=0}^{\infty} \frac{(\lambda)_{2n} t^n}{n!\,(\lambda)_n} F_D^{(r)}\left[(-n, m_i): (\beta_i, \varphi_i); (1-\lambda-2n, m_i); x_1, \ldots, x_r\right]$$

$$= (1-4t)^{1/2}\left[\frac{2}{1+(1-4t)^{1/2}}\right]^{\lambda-1}$$

$$\cdot F_D^{(r)}\left[(\lambda, m_i): (\beta_i, \varphi_i); (\lambda, m_i);\ x_1\left(\frac{2t}{1+(1-4t)^{1/2}}\right)^{m_1},\right.$$

$$\left.\ldots, x_r\left(\frac{2}{1+(1-4t)^{1/2}}\right)^{m_r}\right]. \tag{3.6}$$

Again by making use of the Mittal [6] operational formula

$$\sum_{n=0}^{\infty} \frac{1}{n!} T_{a-n}^{n}\{f(x)\} = (1+x)^{a-1} f[x(1+x)]. \tag{3.7}$$

Proceeding as above we get the generating relation

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1-\lambda)_n}{n!} (t)^n F^{3:1,\ldots,1}_{2:0,\ldots,0}\left[\begin{matrix}(\lambda, m_i), (\lambda, m_i)(-n, m_i); \\ (\beta_i, \varphi_i);\ x_1, \ldots, x_r \\ (\gamma, \psi_i), (\lambda-n, m_i)\end{matrix}\right]$$

$$= (1+t)^{\lambda-1} F_D^{(r)}\left[(\lambda, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i);\right.$$

$$\left. x_1\{-t(1+t)\}^{m_1}, \ldots, x_r\{-t(1+t)\}^{m_r}\right]. \tag{3.8}$$

Next using the Mittal [6] operational formula,

$$\sum_{n=0}^{\infty} \frac{1}{n!} T_{a-2n}^{n}\{f(x)\} = (1+4x)^{-1/2}$$

$$\cdot\left[\frac{2}{1+(1+4x)^{1/2}}\right]^{-a} f\left[\frac{x\left(1+(1+4x)^{1/2}\right)}{2}\right]. \tag{3.9}$$

Putting $a = \lambda$, and

$$f(x) \equiv F_D^{(r)}\left[(\lambda, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i); x_1\left(\tfrac{9}{8}t\right)^{m_1}, \ldots, x_r\left(\tfrac{9}{8}t\right)^{m_r}\right]$$

in (3.9) and assuming that T_k-operates on t alone we get

$$\sum_{n=0}^{\infty} (-)^n \frac{(1-\lambda)_{2n}}{(1-\lambda)_n} \frac{t^n}{n!}$$

$$\cdot F^{4:1,\ldots,1}_{4:0,\ldots,0}\left[\begin{matrix}(-n, m_i), (\lambda, m_i)\left(\frac{\lambda-n+1}{2}, m_i\right)\left(\frac{\lambda^{-n}}{2}, m_i\right); \\ (\beta_i, \varphi_i); x_1, \ldots, x_r \\ (\gamma, \psi_i), \left(\frac{\lambda-2n}{3}, m_i\right), \left(\frac{\lambda-2n+1}{3}, m_i\right), \\ \left(\frac{\lambda-2n+2}{3}, m_i\right)\end{matrix}\right]$$

$$= (1+4t)^{-1/2}\left[\frac{2}{1+(1+4t)^{1/2}}\right]^{-\lambda}$$

$$\cdot F_D^{(r)}\left[(\lambda, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i); \; x_1\left[\tfrac{9}{8}t\left\{1+(1+4t)^{1/2}\right\}\right]^{m_1}, \right.$$

$$\left. \ldots, x_r\left[\tfrac{9}{8}t\left\{1+(1+4t)^{1/2}\right\}\right]^{m_r}\right]. \tag{3.10}$$

Putting $a = \lambda - 1$ and

$$f(x) \equiv F^{(r)}\left[(\lambda, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i); \right.$$

$$\left. x_1\left\{\frac{(2-m)^{2-m}}{(1-m)^{1-m}} t\right\}^{m_1}, \ldots, x_r\left\{\frac{(2-m)^{2-m}}{(1-m)^{1-m}} t\right\}^{m_r}\right]$$

in (2.1) and operating T_k on t alone, we get the general result

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n z^n (\lambda)_{mn}}{n!\,(\lambda)_{(m-1)n}}$$

$$\cdot \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\lambda)_{\Sigma m_i k_i}(\beta_i)_{k_1\varphi_1}\cdots(\beta_r)_{k_r\varphi_r}(-n)_{\Sigma m_i k_i}}{(\gamma)_{k_1\psi_1+\cdots+k_r\psi_r} k_1!\cdots k_r!}$$

$$\cdot \frac{z^{-\Sigma m_i k_i}\left(\frac{\lambda+mn}{1-m}\right)_{\Sigma m_i k_i}\cdots\left(\frac{\lambda+mn-m}{1-m}\right)_{\Sigma m_i k_i} x_1^{k_1}\cdots x_r^{k_r}}{\left(\frac{\lambda+(m-1)n}{2-m}\right)_{\Sigma m_i k_i}\cdots\left(\frac{\lambda+(m-1)n+1-m}{2-m}\right)_{\Sigma m_i k_i}}$$

$$= \frac{(1+v)^{\lambda}}{1-(m-1)v} F_D^{(r)}\left[(\lambda, m_i); (\beta_i, \varphi_i); (\gamma, \psi_i);\right.$$

$$x_1\left\{\frac{(2-m)^{2-m}t(1+v)}{(1-m)^{1-m}}\right\}^{m_1},$$

$$\left.\ldots, x_r\left\{\frac{(2-m)^{2-m}t(1+v)}{(1-m)^{1-m}}\right\}^{m_r}\right], \tag{3.11}$$

where $v = tz(1+v)^m$. Putting $z = 1$ in (3.11) we get

$$\sum_{n=0}^{\infty} (-1)^n \frac{(\lambda)_{mn} t^n}{(\lambda)_{(m-1)n} n!}$$

$$\cdot f^{m:1,\ldots,1}_{m:0,\ldots,0}\left[\begin{matrix}(\lambda, m_i), \left(\frac{\lambda+mn}{1-m}, m_i\right) \cdots \left(\frac{\lambda+mn-m}{1-m}, m_i\right); (\beta_i, \varphi_i); x_1 \cdots x_r \\ (\gamma, \psi_i), \left(\frac{\lambda+(m-1)n}{2-m}, m_i\right) \cdots \left(\frac{\lambda+(m-1)n+1-m}{2-m}, m_i\right)\end{matrix}\right]$$

$$= \frac{(1+v)^{\lambda}}{1-(m-1)v} F_D^{r}\left[(\lambda, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i);\right.$$

$$\left. x_1\left\{\frac{(2-m)^{2-m}}{(1-m)^{1-m}} t(1+v)\right\}^{m_1}, \ldots, x_r\left\{\frac{(2-m)^{2-m}}{(1-m)^{1-m}} t(1+v)\right\}^{m_r}\right]. \tag{3.12}$$

where $v = t(1+v)^m$.

4. Agarwal [8] defined G-function as follows

$$G^{m_1,m_2,n,\nu_1,\nu_2}_{p,q,s,t}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix}(\varepsilon_p) \\ (\gamma_t), (\gamma_t') \\ (e_p) \\ (\beta_q), (\beta_q')\end{matrix}\right] = -\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \varphi(\xi+\eta)\psi(\xi,\eta) x^{\xi} y^{\eta}\, d\xi\, d\eta$$

where

$$\psi(\xi,\eta)=\frac{\Pi_{j=1}^{m_1}\Gamma(\beta_j-\xi)\Pi_{j=1}^{\nu_1}\Gamma(\gamma_j+\xi)\cdot\Pi_{j=1}^{m_2}\Gamma(\beta_j'-\eta)\Pi_{j=1}^{\nu_2}\Gamma(\gamma_j'+\eta)}{\Pi_{j=m_1+1}^{q}\Gamma(1-\beta_j+\xi)\Pi_{j=\nu_1+1}^{t}\Gamma(1-\gamma_j-\xi)\cdot\Pi_{j=m_2+1}^{q}\Gamma(1-\beta_j'+\eta)\Pi_{j=\nu_2+1}^{t}\Gamma(1-\gamma_j'+\eta)},$$

$$\varphi(\xi+\eta)=\frac{\Pi_{j=1}^{n}(1-\varepsilon_j+\xi+\eta)}{\Pi_{j=n+1}^{p}\Gamma(\varepsilon_j-\xi-\eta)\Pi_{j=1}^{s}(e_j+\xi+\eta)} \tag{4.1}$$

and $0 \leqslant m_1 \leqslant q$, $0 \leqslant m_2 \leqslant q$, $0 \leqslant \nu_1 \leqslant t$, $0 \leqslant \nu_2 \leqslant t$, $0 \leqslant n \leqslant p$. Putting $a=\alpha$, $m=\beta+1$ and

$$f(x) \equiv G^{m_1,n_2,n_3,m_2,m_3}_{p_1,[p_2,p_3],q_1,[q_2,q_3]}\left[\begin{array}{c|c} \frac{z_1}{x} & (a_{p_1}) \\ & (1-c_{p_2}),(1-c_{p_3}) \\ & (b_{q_1}) \\ \frac{z_2 y}{x} & (d_{q_2}),(d_{q_3}) \end{array}\right]$$

in (2.1), we get on substituting $z_1=x^2$, $z_2=x$ and $z=t/x$ after operation

$$\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}\,G^{m_1+1,n_2,n_3,m_2,m_3}_{p_1+1,[p_2,p_3],q_1,[q_2,q_3]}\left[\begin{array}{c|c} x & (a_{p_1}),\alpha-n\beta \\ & (1-c_{p_2}),(1-c_{p_3}) \\ & (b_{q_1}),\alpha-n-n\beta \\ y & (d_{q_2}),(d_{q_3}) \end{array}\right]$$

$$=\frac{(1+v)^{1-\alpha}}{(1-\beta v)}\,G^{m_1,n_2,n_3,m_2,m_3}_{p_1,[p_2,p_3],q_1,[q_2,q_3]}\left[\begin{array}{c|c} \frac{x}{1+v} & (a_{p_1}) \\ & (1-c_{p_2}),(1-c_{p_3}) \\ & (b_{q_1}) \\ \frac{y}{1+v} & (d_{q_2})(d_{q_3}) \end{array}\right]$$

where $v=t(1+v)^{\beta+1}$, which is due to Sharma and Abiodun [1].

ACKNOWLEDGEMENT. I am thankful to Dr. R. P. Agarwal for guidance during the preparation of the paper.

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