

ON THE EXISTENCE OF MAXIMAL AND MINIMAL SOLUTIONS FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The existence of maximal and minimal solutions for initial-boundary value problems and the Cauchy initial value problem associated with $Lu = f(x, t, u, \nabla u)$ where L is a second order uniformly parabolic differential operator is obtained by constructing maximal and minimal solutions from all possible lower and all possible upper solutions, respectively. This approach allows f to be highly nonlinear, i.e., f locally Hölder continuous with almost quadratic growth in $|\nabla u|$.

1. Introduction. In this paper we study the existence of maximal and minimal solutions for initial boundary value problems and the Cauchy initial value problem associated with

$$Lu = f(x, t, u, \nabla u), \quad (\text{P})$$

where L is a second order uniformly parabolic differential operator.

In recent years a considerable amount of study has been devoted to establish the existence of solutions for elliptic and parabolic problems provided upper and lower solutions of such problems exist. Much of this work has its basis in the fundamental paper of Nagumo [9] as carried further by Akö [1] and Tomi [17]. Keller [5] and Amann [2] constructed solutions between upper and lower solutions of elliptic problems using a monotone iteration scheme which was possible because of certain one sided Lipschitz continuity assumptions upon the nonlinear terms and because the nonlinearities were assumed gradient independent. (For a recent survey of such results see the paper by Schmitt [16].) Sattinger [15] extended Amann's results (assuming that $f(x, t, u)$ is C^1 with respect to u) to parabolic initial boundary value problems using a similar monotone iteration scheme. This work was subsequently extended to various kinds of problems for both elliptic and parabolic equations by Pao [10], [11] and Puel [13] using either monotone iteration techniques or the theory of monotone operators.

In these papers a minimal solution is produced by starting the iteration scheme with the given lower solution and the maximal solution by commencing the scheme with the upper solution. While this procedure has certain

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computational advantages the permissible class of nonlinearities is restricted to those f described above.

Using a different approach (an approach patterned after methods employed by Akö [1] in the study of Dirichlet boundary value problems for elliptic equations) we demonstrate the existence of maximal and minimal solutions for both initial boundary value and Cauchy initial value problems associated with (P) where a much larger class of nonlinearities f is permissible, e.g., f may depend upon ∇u (though it is necessary to require a Nagumo type growth condition with respect to this variable) and f need not be Lipschitz continuous with respect to u . The maximal and minimal solutions will, respectively, be obtained as the supremum of all possible lower solutions and the infimum of all possible upper solutions where the concept of upper and lower solutions as used here is somewhat more general than that used in the papers quoted above (see definitions below).

In the next section we make precise our assumptions and formulate the problems to be considered. In §3 we state an invariance result from which we deduce the existence of solutions lying between upper and lower solutions while in §4 we state and prove our main result.

2. Definitions and notation. Let \mathbf{R}^n denote n -dimensional real Euclidean space with norm denoted by $|\cdot|$, let Ω be a bounded domain in \mathbf{R}^n whose boundary $\partial\Omega$ is an $n - 1$ dimensional manifold of class $C^{2+\alpha}$, $0 < \alpha < 1$. Let $B^N = \{x \in \mathbf{R}^n: |x| < N\}$, $\pi_T = \Omega \times (0, T)$, $\pi_T^N = B^N \times (0, T)$, $\Gamma_T = (\partial\Omega \times [0, T]) \cup (\bar{\Omega} \times \{0\})$, $\Gamma_T^N = (\partial B^N \times [0, T]) \times (\bar{B}^N \times \{0\})$.

For $u: \bar{D} \times [0, T] \rightarrow \mathbf{R}$ (D will equal either Ω or \mathbf{R}^n , depending on which problem is being considered), define Lu by

$$(Lu)(x, t) = \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t},$$

where $a_{ij}, b_i, c \in C^{\alpha, \alpha/2}(\bar{D} \times [0, T])$, $0 < \alpha < 1$, ($C^{\alpha, \alpha/2}(\cdot)$) shall denote the usual Hölder space of functions $u(x, t)$ which are Hölder continuous with respect to x with exponent α and with respect to t with exponent $\alpha/2$ and which is endowed with the standard Hölder norm, see [6], $1 \leq i, j \leq n$, and $c < 0$. We assume that L is uniformly parabolic, i.e., there exist constants μ, λ , $0 < \lambda < \mu$, such that for all $\xi \in \mathbf{R}^n$ and all $(x, t) \in \bar{D} \times [0, T]$

$$\lambda|\xi|^2 < \sum_{i,j=1}^n a_{ij}(x, t)\xi_i\xi_j \leq \mu|\xi|^2.$$

Let $f: \bar{D} \times [0, T] \times \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$, defined by $(x, t, u, p) \mapsto f(x, t, u, p)$ be a Hölder continuous function with Hölder exponents $\alpha, \alpha/2, \alpha, \alpha$ in the respective variables x, t, u, p .

Given $\psi: \Gamma_T \rightarrow \mathbf{R}$, we consider the initial boundary value problem (IBVP)

$$Lu = f(x, t, u, \nabla u), \quad (x, t) \in \bar{\Omega} \times (0, T], \tag{1}$$

$$u = \psi, \quad (x, t) \in \Gamma_T, \tag{2}$$

where ψ is continuous and may be extended to $\bar{\pi}_T$ so as to belong to $C^{2+\alpha, 1+\alpha/2}(\bar{\pi}_T)$, and satisfies compatibility conditions appropriate to (1), (2).

Given $\varphi: \mathbf{R}^n \rightarrow \mathbf{R}$, we consider the Cauchy initial value problem (CIVP)

$$Lu = f(x, t, u, \nabla u), \quad (x, t) \in \mathbf{R}^n \times (0, T], \tag{3}$$

$$u(x, 0) = \varphi(x), \tag{4}$$

where again φ satisfies appropriate compatibility conditions.

It will be clear from the development to follow that the results will also hold for IBVP's of the second kind.

A continuous function $v: \bar{\pi}_T \rightarrow \mathbf{R}$ is called a *lower solution* of (1), (2) in case

$$v(x, t) \leq \psi(x, t), \quad (x, t) \in \Gamma_T, \tag{5}$$

and if for every $(x_0, t_0) \in \pi_T$ there exists an open neighborhood U of (x_0, t_0) and a finite set of functions $\{v_r\}_{1 \leq r \leq s} \subseteq C^{2,1}(\bar{U} \cap \pi_T)$ such that

$$Lv_r \geq f(x, t, v_r, \nabla v_r), \quad (x, t) \in \bar{U} \cap \pi_T, 1 \leq v \leq s, \tag{6}$$

and

$$v(x, t) = \max_{1 \leq r \leq s} v_r(x, t), \quad (x, t) \in \bar{U} \cap \pi_T. \tag{7}$$

If in the above definition the inequality signs in (5) and (6) are reversed and in (7) max is replaced by min then v is called an upper solution of (1), (2).

One defines lower solutions for (3), (4) similarly, by replacing π_T everywhere by $\mathbf{R}^n \times (0, T)$ and (5) by

$$v(x, 0) \leq \varphi(x), \quad x \in \mathbf{R}^n. \tag{8}$$

In analogous manner the concept of an upper solution for (3), (4) is defined.

If in any of the above definitions the inequalities occurring are replaced by strict inequalities the lower (resp. upper) solution will be called a strict lower (resp. strict upper) solution.

3. Invariance and existence. The first result needed in the development to follow is a known invariance result, a more general version of which appears in [3]. In the form given it is a Nagumo-Westphal type lemma.

LEMMA 1. *Let α, β be strict lower and upper solutions of (1), (2) with $\alpha(x, t) < \beta(x, t), (x, t) \in \bar{\pi}_T$. If u is a solution of (1), (2), then $\alpha(x, t) < u(x, t) < \beta(x, t), (x, t) \in \bar{\pi}_T$.*

PROOF. Let $u \in C^{2,1}(\bar{\pi}_T)$ be a solution of (1), (2) and assume

$$u(\bar{\pi}_T) \not\subset \{v: \alpha(x, t) < v < \beta(x, t), (x, t) \in \bar{\pi}_T\} \stackrel{\text{def}}{=} (\alpha, \beta).$$

We then may assume (other cases may be argued analogously) that there exists $(x_0, t_0) \in \pi_T$ such that $u(x_0, t_0) = \alpha(x_0, t_0)$ and $\alpha(x, t) < u(x, t), (x, t) \in \Omega \times [0, t_0)$. There exists a neighborhood U of (x_0, t_0) and functions $\{\alpha_r\}_{1 \leq r \leq s}$ satisfying (6) and (7) with strict inequalities holding in (6). Let $\alpha_k(x_0, t_0) = \alpha(x_0, t_0)$, and put $v(x, t) = u(x, t) - \alpha_k(x, t)$, then $v(x, t) > 0$,

$(x, t) \in (\Omega \times [0, t_0]) \cap U, v_t(x_0, t_0) < 0, \nabla v(x_0, t_0) = 0$ and

$$\sum_{i,j=1}^n a_{ij}(x_0, t_0) \frac{\partial v(x_0, t_0)}{\partial x_i \partial x_j} > 0$$

and thus $(Lv)(x_0, t_0) \geq 0$. But

$$\begin{aligned} (Lv)(x_0, t_0) &= (Lu)(x_0, t_0) - (L\alpha_k)(x_0, t_0) \\ &= f(x_0, t_0, u, \nabla u) - (L\alpha_k)(x_0, t_0) \\ &= f(x_0, t_0, \alpha_k(x_0, t_0), \nabla \alpha_k(x_0, t_0)) - (L\alpha_k)(x_0, t_0) \\ &< 0, \text{ by hypothesis.} \end{aligned}$$

We have thus arrived at a contradiction.

Using this invariance result we derive via an existence result from [3] the following existence theorem. A similar result is contained in Puel [13, Theorem 3.1, p. 97], though with a more complicated proof.

THEOREM 2. *Let α, β be lower and upper solutions of (1), (2) respectively with $\alpha(x, t) < \beta(x, t), (x, t) \in \bar{\pi}_T$. Let there exist a positive continuous nondecreasing function $\Phi: [0, \infty) \rightarrow (0, \infty)$ such that $s^2/\Phi(s) \rightarrow \infty$ as $s \rightarrow \infty$ and*

$$|f(x, t, u, p)| \leq \Phi(|p|),$$

$\alpha(x, t) < u < \beta(x, t), (x, t) \in \bar{\pi}_T$.

Then the IBVP (1), (2) has a solution $u \in C^{2,1}(\bar{\pi}_T)$ with $\alpha(x, t) < u(x, t) < \beta(x, t), (x, t) \in \bar{\pi}_T$.

PROOF. Define \tilde{f} in the following way

$$\tilde{f}(x, t, u, p) = \begin{cases} f(x, t, \beta(x, t), p) + (u - \beta(x, t)), & \text{if } u > \beta(x, t), \\ f(x, t, u, p), & \text{if } \alpha(x, t) \leq u \leq \beta(x, t), \\ f(x, t, \alpha(x, t), p) + (u - \alpha(x, t)), & \text{if } u < \alpha(x, t). \end{cases}$$

Choose constants a and $b, a < 0 < b$, such that $a < \alpha(x, t) < \beta(x, t) < b, (x, t) \in \bar{\pi}_T$ and

$$\tilde{f}(x, t, a, 0) < 0 < \tilde{f}(x, t, b, 0). \tag{9}$$

Using Theorem 2 of [3] we may conclude that the equation

$$Lu = \tilde{f}(x, t, u, \nabla u), \quad (x, t) \in \bar{\Omega} \times (0, T], \tag{10}$$

has a solution u satisfying (2) and $a < u(x, t) < b$. Let $\epsilon > 0$ be given. It follows from the definition of \tilde{f} that $\alpha(x, t) - \epsilon$ and $\beta(x, t) + \epsilon$ are strict lower and upper solutions of (1), (2), respectively. Thus by Lemma 1 we have that $\alpha(x, t) - \epsilon < u(x, t) < \beta(x, t) + \epsilon, (x, t) \in \bar{\pi}_T$. Since $\epsilon > 0$ was arbitrary, we conclude that in fact $\alpha(x, t) < u(x, t) < \beta(x, t), (x, t) \in \bar{\pi}_T$, and thus u is a solution of (1), (2).

From the above existence theorem for IBVP's for (1), (2) we can easily derive the following existence theorem for the CIVP (3), (4).

THEOREM 3. *Let α and β be lower and upper solutions, respectively, of (3), (4). For each $N > 0$ let there exist a function Φ_N having the properties of Φ in Theorem 2 on the set $\bar{\pi}_T^N$. Then the CIVP (3), (4) has a solution $u \in C^{2,1}(\mathbf{R}^n \times [0, T])$ such that $\alpha(x, t) \leq u(x, t) \leq \beta(x, t)$, $(x, t) \in \mathbf{R}^n \times [0, T]$.*

PROOF. For each positive integer N , there exists a solution u_N of the IBVP

$$\begin{cases} Lu = f(x, t, u, \nabla u), & (x, t) \in B^N \times (0, T], \\ u_N(x, 0) = \varphi(x), & x \in B^N, \\ u_N(x, t) = \theta_N(x, t), & (x, t) \in \partial B^N \times [0, T], \end{cases} \quad (11)$$

where θ_N is a sufficiently smooth function which satisfies

$$\theta_N(x, 0) = \varphi(x), \quad \alpha(x, t) \leq \theta_N(x, t) \leq \beta(x, t),$$

$(x, t) \in \bar{\Gamma}_T^N$, as follows easily by an application of Theorem 2. Using standard estimates and a diagonalization process one obtains a subsequence of $\{u_N\}_{N=1}^\infty$ which converges uniformly on compact subsets of $\mathbf{R}^n \times [0, T]$ to a solution of the CIVP (3), (4).

4. Maximal and minimal solutions. In this section we establish the existence of maximal and minimal solutions of the IBVP (1), (2) and the CIVP (3), (4). Mlak [7] has obtained such results for special classes of nonlinearities for IBVP (1), (2).

A solution \bar{u} of the IBVP (1), (2) [CIVP (3), (4)] is a maximal solution relative to a given pair of lower and upper solutions α and β with $\alpha(x, t) < \beta(x, t)$, $(x, t) \in \bar{\pi}_T$ [$(x, t) \in \mathbf{R}^n \times [0, T]$], if $\alpha(x, t) < \bar{u}(x, t) < \beta(x, t)$, and if u is any other such solution then $u(x, t) \leq \bar{u}(x, t)$ for $(x, t) \in \bar{\pi}_T$ [$(x, t) \in \mathbf{R}^n \times [0, T]$]. Minimal solutions are defined analogously.

THEOREM 4. *Let the hypotheses of Theorem 2 hold. Then the IBVP (1), (2) has a maximal solution u_{\max} and a minimal solution u_{\min} relative to the pair α, β .*

PROOF. Let $\mathcal{L} = \{v: \bar{\pi}_T \rightarrow \mathbf{R}, \alpha(x, t) \leq v(x, t) \leq \beta(x, t), (x, t) \in \bar{\pi}_T, v \text{ is a lower solution of (1), (2)}\}$, and let $\mathcal{U} = \{w: \bar{\pi}_T \rightarrow \mathbf{R}, \alpha(x, t) \leq w(x, t) \leq \beta(x, t), (x, t) \in \bar{\pi}_T, w \text{ is an upper solution of (1), (2)}\}$. Define u_{\max} and u_{\min} by

$$u_{\max}(x, t) = \sup\{v(x, t): v \in \mathcal{L}\}, \quad u_{\min}(x, t) = \inf\{w(x, t): w \in \mathcal{U}\}.$$

We shall establish that u_{\max} , thus defined, is a maximal solution, that u_{\min} is a minimum solution will follow in much the same way.

Let $\{(x_N, t_N)\}_{N=1}^\infty$ be a countable dense subset of $\bar{\pi}_T$ and for $N = 1, 2, \dots$, let $\{v_{N,m}\}_{m=1}^\infty$ be a sequence of lower solutions such that

$$\lim_{m \rightarrow \infty} v_{N,m}(x_N, t_N) = u_{\max}(x_N, t_N).$$

Let $\alpha_1(x, t) = v_{1,1}(x, t)$; it follows from Theorem 2 that there exists a solution

u_1 of (1), (2) such that $\alpha_1(x, t) \leq u_1(x, t) \leq \beta(x, t)$, $(x, t) \in \bar{\pi}_T$. Let α_2 be defined by

$$\alpha_2(x, t) = \max\{u_1(x, t), v_{1,2}(x, t), v_{2,2}(x, t)\},$$

then α_2 is a lower solution of (1), (2) and by a further application of Theorem 2 there exists a solution u_2 of (1), (2) such that $\alpha_2(x, t) \leq u_2(x, t) \leq \beta(x, t)$, $(x, t) \in \bar{\pi}_T$. Inductively let α_{i+1} be the lower solution of (1), (2) defined by

$$\alpha_{i+1}(x, t) = \max\{u_i(x, t), v_{1,i}(x, t), \dots, v_{i,i}(x, t)\},$$

where u_i is a solution of (1), (2) such that $\alpha_i(x, t) \leq u_i(x, t) \leq \beta(x, t)$, $(x, t) \in \bar{\pi}_T$. We thus have, inductively, obtained a sequence of solutions $\{u_i\}_{i=1}^\infty$ of (1), (2) such that $\alpha(x, t) \leq u_1(x, t) \leq \dots \leq u_i(x, t) \leq \dots \leq \beta(x, t)$, $(x, t) \in \bar{\pi}_T$. Arguments similar to those used in the proof of Theorem 2 of [3] show that the sequence $\{u_i\}_{i=1}^\infty$ will converge to a solution u of (1), (2). Furthermore it is clear that

$$\lim_{i \rightarrow \infty} u_i(x_N, t_N) = u_{\max}(x_N, t_N)$$

for $N = 1, 2, \dots$. Hence $u(x_N, t_N) \geq v(x_N, t_N)$, for all $v \in \mathcal{L}$ and $N = 1, 2, \dots$ and thus since $\{(x_N, t_N)\}_{N=1}^\infty$ is dense in $\bar{\pi}_T$ it follows that $u(x, t) \geq v(x, t)$ for all $(x, t) \in \bar{\pi}_T$, $v \in \mathcal{L}$. Thus $u(x, t) = u_{\max}(x, t)$, $(x, t) \in \bar{\pi}_T$ and consequently u_{\max} is a maximal solution of (1), (2).

Using a similar argument and Theorem 3 one obtains the following result.

THEOREM 5. *Let the hypotheses of Theorem 3 hold. Then the CIVP (3), (4) has a maximal solution u_{\max} and a minimal solution u_{\min} relative to the pair α, β .*

REMARK. It follows from Theorem 7 of [3] that the set of solutions u of (1), (2) with $u_{\min}(x, t) \leq u(x, t) \leq u_{\max}(x, t)$, $(x, t) \in \bar{\pi}_T$, is a continuum in $C^{1,0}(\bar{\pi}_T)$.

REMARK. The proof of Theorem 4 is based on a method due to Akö [1].

EXAMPLES. 1. The following example is similar to one given by Prodi [12] and Redheffer and Walter [14] to illustrate nonuniqueness for IBVP (1), (2). Consider

$$\begin{aligned} u_{xx} - u_t &= -(u + (\cos x \cdot u)^{1/2}), & (x, t) \in (-\pi/2, \pi/2) \times (0, 1), \\ u(x, 0) &= 0, & x \in (-\pi/2, \pi/2), \\ u(-\pi/2, t) &= 0 = u(\pi/2, t), & t \in (0, 1). \end{aligned}$$

Then $\alpha(x, t) \equiv 0$ and $\beta(x, t) = t^2$ are lower and upper solutions, respectively, on $[-\pi/2, \pi/2] \times [0, 1]$. The family $\{u(x, t; t_0)\}$ given by

$$u(x, t; t_0) = \begin{cases} 0, & 0 \leq t \leq t_0, \\ \frac{(t - t_0)^2 \cos x}{4}, & t_0 \leq t \leq 1, \end{cases}$$

are solutions. By Theorem 4, u_{\max} and u_{\min} exist between α and β on $[-\pi/2, \pi/2] \times [0, 1]$.

2. For the CIVP: $\Delta u - u_t = -u^{1-\gamma}$, $\gamma \in (0, 1)$ with $u(x, 0) = 0$, $\beta(x, t) = (\gamma t)^{1/\gamma}$ and $\alpha(x, t) = 0$ are upper and lower solutions on $\mathbf{R}^n \times [0, T]$. By Theorem 5, u_{\max} and u_{\min} exist.

3. The next example is similar to one given by Mlak [8] for the IBVP. Consider $u_{xx} - u_t = f(x, u)$, $(x, t) \in \mathbf{R} \times (0, \pi/2)$, $u(x, 0) = \cos x$, $x \in \mathbf{R}$, where

$$f(x, u) = \begin{cases} -u + \operatorname{sgn}(\cos x) \sqrt{\cos^2 x - u^2}, & |u| \leq \cos x, \\ -u, & |u| \geq \cos x. \end{cases}$$

Then $\beta(x, t) = 3e^t - 1$ and $\alpha(x, t) = -3e^t + 1$ are upper and lower solutions, respectively. The family $\{u(x, t; t_0)\}$ given by

$$u(x, t; t_0) = \begin{cases} \cos x, & 0 \leq t \leq t_0, \\ \cos(t - t_0) \cos x, & t_0 \leq t \leq \pi/2, \end{cases}$$

are solutions of the given CIVP. By Theorem 5, u_{\max} and u_{\min} exist.

REMARK. Prodi [12, p. 37] has noted that if f satisfies a Hölder condition and a Nagumo growth condition with respect to p , then locally upper and lower solutions can be constructed. With this observation any problem (IBVP or CIVP) which has nonuniqueness locally has distinct maximal and minimal solutions respectively by our Theorems 4 and 5.

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