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PARACOMPACTNESS, METACOMPACTNESS, AND SEMI-OPEN COVERS

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ABSTRACT. Paracompactness and metacompactness are characterized in terms of locally finite and point-finite semi-open refinements of open covers. It follows from one of these characterizations that a continuous image of a paracompact space under a pseudo-open and compact mapping is metacompact.

1. On semi-open covers. For the meaning of concepts used without definition in this paper, see [3]; note, however, that we do not require paracompact spaces or metacompact spaces to satisfy any separation axioms.

Throughout the following, X denotes a topological space. Let $\mathcal{L}$ be a cover of X. For each $x \in X$, we let $(\mathcal{L})_x = \{L \in \mathcal{L} | x \in L\}$. Note that we have $\mathrm{St}(x, \mathcal{L}) = \bigcup(\mathcal{L})_x$ for each $x \in X$. If the set $\mathrm{St}(x, \mathcal{L})$ is a neighborhood of x for each $x \in X$, then we say that $\mathcal{L}$ is a *semi-open cover* of X. For some properties of semi-open covers, see [6]. When $\mathcal{N}$ is a cover of X, we say that $\mathcal{N}$ is an *F-refinement* of the cover $\mathcal{L}$ if each set $N \in \mathcal{N}$ is contained in some finite union of sets of the family $\mathcal{L}$.

LEMMA 1.1. *A locally finite semi-open cover of a topological space has a locally finite closed F-refinement.*

PROOF. Let $\mathcal{L}$ be a locally finite and semi-open cover of X. For each subfamily $\mathcal{L}'$ of $\mathcal{L}$, let $K(\mathcal{L}') = \mathrm{Cl}(\bigcap \mathcal{L}') \sim \mathrm{Int}(\bigcup(\mathcal{L} \sim \mathcal{L}'))$. Note that if $\mathcal{L}'$ is infinite, then $K(\mathcal{L}') = \emptyset$. For each $\mathcal{L}' \subset \mathcal{L}$, we have $K(\mathcal{L}') \subset \bigcup \mathcal{L}'$. To see this, let $x \in K(\mathcal{L}')$. Then $x \notin \mathrm{Int}(\bigcup(\mathcal{L} \sim \mathcal{L}'))$ and it follows, since $x \in \mathrm{Int}(\bigcup(\mathcal{L})_x)$, that we have $(\mathcal{L})_x \cap \mathcal{L}' \neq \emptyset$, in other words, $x \in \bigcup \mathcal{L}'$.

Since $x \in K((\mathcal{L})_x)$ for each $x \in X$, it follows from the foregoing that the closed family $\mathcal{K} = \{K(\mathcal{L}') | \mathcal{L}' \subset \mathcal{L}\}$ is an F-refinement of $\mathcal{L}$. To show that $\mathcal{K}$ is locally finite, let $x \in X$. Since $\mathcal{L}$ is locally finite, the subfamily $\mathcal{L}^* = \{L \in \mathcal{L} | x \in \bar{L}\}$ is finite and the open set $O = X \sim \mathrm{Cl}(\bigcup(\mathcal{L} \sim \mathcal{L}^*))$ contains x. If $\mathcal{L}' \subset \mathcal{L}$ and $K(\mathcal{L}') \cap O \neq \emptyset$, then $[\mathrm{Cl}(\bigcap \mathcal{L}')] \cap O \neq \emptyset$ and hence $(\bigcap \mathcal{L}') \cap O \neq \emptyset$. It follows that if $K(\mathcal{L}') \cap O \neq \emptyset$, then $\mathcal{L}' \subset \mathcal{L}^*$; hence the neighborhood O of x intersects only finitely many sets of the family $\mathcal{K}$. □

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In the next section, we use the result of Lemma 1.1 to derive a characterization of paracompactness in terms of the existence of locally finite semi-open refinements of certain open covers. It is not difficult to see that a point-finite semi-open (or even open) cover of a topological space does not always have a point-finite semi-open F-refinement by closed sets (see the remark following Theorem 2.2 below). To be able to characterize metacompactness in terms of point-finite semi-open refinements, we show that the existence of such refinements implies the existence of certain open refinements.

Let $\mathcal{L}$ and $\mathcal{N}$ be covers of X. We say that $\mathcal{N}$ is a *point-wise W-refinement* of $\mathcal{L}$ if for each $x \in X$, there exists a finite subfamily $\mathcal{L}'$ of $\mathcal{L}$ such that for each $N \in (\mathcal{N})_x$, we have $N \subset L$ for some $L \in \mathcal{L}'$.

LEMMA 1.2. *If an open cover of a topological space has a point-finite semi-open refinement, then the cover has an open point-wise W-refinement.*

PROOF. Let $\mathcal{L}$ be a point-finite semi-open refinement of an open cover $\mathcal{U}$ of X. For each $L \in \mathcal{L}$, let $U(L) \in \mathcal{U}$ be such that $L \subset U(L)$. For each $x \in X$, denote by $\mathcal{U}(x)$ the finite subfamily $\{U(L)|L \in (\mathcal{L})_x\}$ of $\mathcal{U}$ and denote by $V(x)$ the open neighborhood $[\text{Int St}(x, \mathcal{L})] \cap [\bigcap \mathcal{U}(x)]$ of x. We show that the open cover $\mathcal{V} = \{V(x)|x \in X\}$ of X is a point-wise W-refinement of the cover $\mathcal{U}$. Let $x \in X$ and let $y \in X$ be such that $x \in V(y)$. Then $x \in \text{St}(y, \mathcal{L})$ and hence there exists $L \in \mathcal{L}$ such that $x \in L$ and $y \in L$. For the set $U(L)$, we have $U(L) \in \mathcal{U}(x)$ and $V(y) \subset U(L)$. We have shown that for each $V \in (\mathcal{V})_x$, we have $V \subset U$ for some member U of the finite subfamily $\mathcal{U}(x)$ of $\mathcal{U}$. □

Our remaining lemmas deal with the preservation of the property of semi-openness in certain topological operations.

LEMMA 1.3. *Let $\mathcal{L}$ be a point-finite semi-open cover of X and for each $L \in \mathcal{L}$, let $\mathcal{N}(L)$ be a (point-finite) semi-open cover of the subspace L of X. Then the family $\mathcal{N} = \bigcup\{\mathcal{N}(L)|L \in \mathcal{L}\}$ is a (point-finite) semi-open cover of X.*

PROOF. It is easily seen that the family $\mathcal{N}$ is point-finite if the families $\mathcal{L}$ and $\mathcal{N}(L)$, $L \in \mathcal{L}$, are all point-finite. To show that $\mathcal{N}$ is a semi-open cover of X, let $x \in X$. For each $L \in (\mathcal{L})_x$, the set $\text{St}(x, \mathcal{N}(L))$ is a neighborhood of x in the subspace L of X and it follows that there exists a neighborhood $O(L)$ of x in X such that $O(L) \cap L = \text{St}(x, \mathcal{N}(L))$. The set $O = [\text{St}(x, \mathcal{L})] \cap [\bigcap\{O(L)|L \in (\mathcal{L})_x\}]$ is a neighborhood of x in X. We show that $O \subset \text{St}(x, \mathcal{N})$. Let $y \in O$. Then $y \in \text{St}(x, \mathcal{L})$ and hence there exists $L \in (\mathcal{L})_x$ such that $y \in L$. But then we have $y \in L \cap O(L) = \text{St}(x, \mathcal{N}(L)) \subset \text{St}(x, \mathcal{N})$. Hence $O \subset \text{St}(x, \mathcal{N})$ and the set $\text{St}(x, \mathcal{N})$ is a neighborhood of x. □

A mapping f from X onto a topological space Y is called *pseudo-open* ([**1**]; in [**8**] these were called P_1-mappings) provided that for each $y \in Y$, whenever U is a neighborhood of the set $f^{-1}\{y\}$ in the space X, then the set $f(U)$ is a neighborhood of the point y in the space Y.

LEMMA 1.4. *Let X and Y be topological spaces, let $\mathfrak{L}$ be a semi-open cover of X and let f be a pseudo-open mapping from X onto Y. Then the family $\mathfrak{N} = \{f(L) | L \in \mathfrak{L}\}$ is a semi-open cover of Y.*

PROOF. The conclusion follows directly from the definitions, since we have $\mathrm{St}(y, \mathfrak{N}) = f(\mathrm{St}(f^{-1}\{y\}, \mathfrak{L}))$ for every $y \in Y$. □

2. On paracompactness and metacompactness. We start by characterizing paracompactness. Recall that a family $\mathfrak{N}$ of sets is *monotone* provided that the relation $\subset$ of set inclusion is a linear order on $\mathfrak{N}$.

THEOREM 2.1. *A topological space is paracompact if, and only if, every monotone open cover of the space has a locally finite semi-open refinement.*

PROOF. Necessity of the condition is obvious. To prove sufficiency, assume that every monotone open cover of X has a locally finite semi-open refinement. For every cardinal number $\underline{k}$, denote by $P(\underline{k})$ the following proposition: if $\mathfrak{U}$ is an open cover of X with $|\mathfrak{U}| = \underline{k}$, then $\mathfrak{U}$ has a locally finite closed F-refinement. We observe that $P(\underline{k})$ is trivially true for $\underline{k}$ finite and we use transfinite induction to show that $P(\underline{k})$ holds in general. Let $\underline{k}$ be an infinite cardinal number such that $P(\underline{h})$ holds for every $\underline{h} < \underline{k}$. To show that $P(\underline{k})$ holds, let $\mathfrak{U}$ be an open cover of X with $|\mathfrak{U}| = \underline{k}$. We represent $\mathfrak{U}$ in the form $\mathfrak{U} = \{U_\alpha | \alpha < \gamma\}$, where γ is the initial ordinal ordinal corresponding to the cardinal $\underline{k}$. For each $\alpha < \gamma$, let $V_\alpha = \bigcup_{\beta < \alpha} U_\beta$. Then the family $\mathfrak{V} = \{V_\alpha | \alpha < \gamma\}$ is a monotone open cover of X. The cover $\mathfrak{V}$ has a locally finite semi-open refinement and it follows from Lemma 1.1 that $\mathfrak{V}$ has a locally finite closed F-refinement, say $\mathfrak{K}$. Let K be a member of the family $\mathfrak{K}$. Then K is contained in some finite union of sets of the family $\mathfrak{V}$ and it follows, since $\mathfrak{V}$ is a monotone family, that K is contained in some set of $\mathfrak{V}$. Let $\alpha(K) < \gamma$ be such that $K \subset V_{\alpha(K)}$. The family $\mathfrak{W}(K) = \{X \sim K\} \cup \{U_\alpha | \alpha < \alpha(K)\}$ is an open cover of X and we have $|\mathfrak{W}(K)| < \underline{k}$. By the induction assumption, $\mathfrak{W}(K)$ has a locally finite closed F-refinement, say $\mathfrak{F}(K)$. For every $K \in \mathfrak{K}$, the family $\mathfrak{F}'(K) = \{F \cap K | F \in \mathfrak{F}(K)\}$ is a locally finite closed cover of the subspace K of X. Since $\mathfrak{K}$ is a locally finite and closed cover of X, it follows that the family $\mathfrak{F} = \bigcup\{\mathfrak{F}'(K) | K \in \mathfrak{K}\}$ is also a locally finite and closed cover of X. It is easily seen that every set of the family $\mathfrak{F}$ is contained in some finite union of sets of the cover $\mathfrak{U}$; hence $\mathfrak{F}$ is an F-refinement of $\mathfrak{U}$. We have shown that $P(\underline{k})$ holds. This completes the induction.

It follows from the foregoing that every open cover of X has a locally finite closed F-refinement. Since a directed cover (see [7]) is an F-refinement of itself, it follows that every directed open cover of X has a locally finite closed refinement. By Corollary 6 of [7], the space X is paracompact. □

Theorem 2.1 generalizes some results of J. Mack [7].

It is not known if the existence of point-finite semi-open refinements for all monotone open covers of a topological space is sufficient for the space to be metacompact; however, we have the following result:

THEOREM 2.2. *A topological space is metacompact if, and only if, every open cover of the space has a point-finite semi-open refinement.*

PROOF. Necessity is obvious and sufficiency follows directly from Lemma 1.2 and the result of J. M. Worrell Jr. that a topological space is metacompact if every open cover of the space has an open point-wise W-refinement [11]. □

Using Theorem 2.2 and the technique used in the proof of Theorem 2.1, it can be shown that a topological space is metacompact if every monotone open cover of the space has a point-finite semi-open closed refinement; for normal spaces this condition is also necessary, since every point-finite open cover of a normal space has an open shrinking (see e.g. [3, Theorem 1.5.18]). In general, however, monotone open covers of metacompact spaces do not necessarily have point-finite semi-open closed refinements (to see this, consider the monotone open cover $\{X \sim \{1/k | k \geqslant n\} | n \in \mathbf{N}\}$ of the space X of Example 5.3.4 of [3]).

We close this paper with two corollaries to Theorem 2.2.

COROLLARY 2.3. *A topological space is metacompact if it has a point-finite semi-open cover such that every set of the cover is contained in some metacompact subspace of the space.*

PROOF. Assume that X has a point-finite semi-open cover $\mathcal{L}$ such that for each $L \in \mathcal{L}$, there exists a metacompact subspace $M(L)$ of X such that $L \subset M(L)$. To show that X is metacompact, let $\mathcal{U}$ be an open cover of X. For each $L \in \mathcal{L}$, the family $\mathcal{U}|M(L) = \{U \cap M(L) | U \in \mathcal{U}\}$ is an open cover of the subspace $M(L)$ of X and hence there exists a point-finite open cover $\mathcal{N}(L)$ of the subspace $M(L)$ such that $\mathcal{N}(L)$ is a refinement of $\mathcal{U}|M(L)$. For each $L \in \mathcal{L}$, the family $\mathcal{N}'(L) = \{N \cap L | N \in \mathcal{N}(L)\}$ is a point-finite open cover of the subspace L of X. It follows from Lemma 1.3 that the family $\mathcal{N} = \cup\{\mathcal{N}'(L) | L \in \mathcal{L}\}$ is a point-finite semi-open cover of X and it is easily seen that the family $\mathcal{N}$ is a refinement of the cover $\mathcal{U}$. By the foregoing and Theorem 2.2, the space X is metacompact. □

R. E. Hodel has shown in [5] that the Locally Finite Sum Theorem holds for metacompactness; since a locally finite closed cover is semi-open, Corollary 2.3 generalizes Hodel's result. It is well known that the analogue of the result of Corollary 2.3 for paracompactness is false; for instance, in [4] there is an example of a nonparacompact Moore space that is the union of two open metrizable subspaces. However, the Locally Finite Sum Theorem also holds for paracompactness ([10]; for regular spaces, [9]).

In [2], A. V. Arhangel'skiĭ proved that a continuous image of a metrizable space under a pseudo-open and compact mapping is metacompact and he asked whether this result remains true if "metrizable" is replaced by "paracompact"; the following result shows that the answer to this question is in the affirmative.

COROLLARY 2.4. *A continuous image of a paracompact space under a pseudo-open and compact mapping is metacompact.*

PROOF. Let X be a paracompact space, and let f be a pseudo-open, compact and continuous mapping from X onto a topological space Y. To show that Y is metacompact, let $\mathcal{U}$ be an open cover of Y. Then the family $\mathcal{O} = \{f^{-1}(U) | U \in \mathcal{U}\}$ is an open cover of X. Let $\mathcal{V}$ be a locally finite open refinement of $\mathcal{O}$ and let $\mathcal{M} = \{f(V) | V \in \mathcal{V}\}$. It is easily seen that $\mathcal{M}$ is a refinement of the cover $\mathcal{U}$ of Y and it follows from Lemma 1.4 that $\mathcal{M}$ is a semi-open cover of Y. We also see that the family $\mathcal{M}$ is point-finite since for each $y \in Y$, the compact subset $f^{-1}\{y\}$ of X meets only finitely many sets of the locally finite family $\mathcal{V}$. We have shown that every open cover of Y has a point-finite semi-open refinement. By Theorem 2.2, the space Y is metacompact. □

Note that the above proof can be modified so as to yield the following result: a continuous image of a metacompact space under a pseudo-open and finite-to-one mapping is metacompact.

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