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ISOGENY RESTRICTIONS OF IRREDUCIBLE ADMISSIBLE REPRESENTATIONS ARE FINITE DIRECT SUMS OF IRREDUCIBLE ADMISSIBLE REPRESENTATIONS[1]

ALLAN J. SILBERGER

ABSTRACT. This paper proves that the isogeny restriction of an irreducible admissible representation of a connected reductive algebraic p-adic group decomposes as a finite direct sum of irreducible admissible representations.

Let $\tilde{G} \xrightarrow{\tilde{\phi}} \tilde{G}^* \to 1$ be a central Ω-isogeny of connected reductive Ω-groups, where Ω is a nonarchimedean local field. Then $\tilde{\phi}$ induces a continuous homomorphism ϕ: $G \to G^*$ of the corresponding groups of Ω-points and $\phi(G)$ is a closed subgroup of G^* such that $G^*/\phi(G)$ is compact and abelian.

The purpose of this note is to prove the following:

THEOREM. *Let π^* be an irreducible admissible representation of G^*. Then $\pi = \pi^* \circ \phi$ decomposes as a finite direct sum of irreducible admissible representations of G.*

We thank the referee for pointing out that our original proof of the above theorem could be shortened and simplified.

The proof. In the following capital letters denote groups of Ω-points of Ω-groups; a tilde over the letter indicates the algebraic group.

LEMMA 1. *If π^* is supercuspidal, then π is supercuspidal and finitely generated.*

PROOF. Since π^* is smooth and irreducible and since $G^*/\phi(G)$ is compact, π is finitely generated. Since every unipotent element of G^* lies in the image of ϕ and since π^* is supercuspidal, π is supercuspidal (i.e., admissible, too, as follows from [**1**, Theorem 6] and [**2**, Lemma 1.11.4]).

LEMMA 2. *Let π^* be a finitely generated admissible representation of G^*. Then π is also finitely generated and admissible.*

PROOF. If $0 \to V_1 \to V_2 \to V_3 \to 0$ is an exact sequence of smooth G^*-modules, then V_2 is admissible if and only if V_1 and V_3 are. An admissible

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representation is finitely generated if and only if it has finite length [2, Theorem 5.4.1.6]. Thus, it suffices to consider the case in which π^* is irreducible.

In this case, by a theorem of Jacquet, one can find a p-subgroup $P^* = M^*N^*$ of G^* and an irreducible supercuspidal representation σ^* of M^* such that π^* is imbedded in $\mathrm{Ind}_{P^*}^{G^*}\sigma^*$. Let $P = \phi^{-1}(P^*)$. Then P is a p-subgroup of G. There is a Levi decomposition $\tilde{P} = \tilde{M}\tilde{N}$ for $\tilde{P}$ such that $\tilde{\phi}|\tilde{P}$ induces an isogeny $\tilde{M} \overset{\tilde{\phi}_M}{\to} \tilde{M}^* \to 1$. Since representatives for $G^*/\phi(G)$ may be chosen in M^* and since both G^* and G have Iwasawa decompositions, it is clear that $(\mathrm{Ind}_{P^*}^{G^*}\sigma^*) \circ \phi = \mathrm{Ind}_P^G\sigma$, where $\sigma = \sigma^* \circ \phi_M$. Since the functor Ind_P^G sends admissible M-modules of finite type to admissible G-modules of finite type [2, Theorem 5.4.4.1], this lemma follows from the preceding one.

Lemma 3. *If π^* is an irreducible admissible representation of G^*, then π is completely reducible.*

Proof. By Lemma 2, π is admissible of finite type. Let π_1 be any irreducible subrepresentation of π. Then, for any $g \in G^*$, $\pi^*(g)$ intertwines π_1 with an irreducible subrepresentation ${}^g\pi_1$ of π, where

$$ {}^g\pi_1(x) = \pi_1(gxg^{-1}) \quad \text{for all } x \in G. $$

Thus, π^* stabilizes the maximum completely reducible subrepresentation of π, so π is itself completely reducible.

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Department of Mathematics, Cleveland State University, Cleveland, Ohio 44115