PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 3, March 1979

REFLEXIVE MODULES OVER CERTAIN DIFFERENTIAL POLYNOMIAL RINGS

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ABSTRACT. Let K be a commutative Noetherian integral domain with a derivation d and let $R = K[x, d]$. When K is quasi-normal and d is suitably restricted we shall give several equivalent conditions for an R-module to be reflexive. The relations between K and R in the context of reflexivity are also investigated.

1. Introduction and preliminaries. Our aim in this note is to give necessary and sufficient conditions for modules over a class of Noetherian differential polynomial domains to be reflexive. Let K be a ring with a derivation d, and let $R = K[x, d]$ denote the ring of differential polynomials over K in the derivation d. As abelian groups $R = K[x]$, but multiplication in R is given by the rule $kx = xk + d(k)$. The study of reflexive R-modules is naturally closely related to the properties of K, and so for K we shall take a commutative Noetherian domain over which the reflexive modules are known. So far, *quasi-normal* domains [7] seem to be the most general commutative Noetherian domains over which the reflexive modules are characterized. In view of [3, Theorem 3.8] a commutative Noetherian domain K is quasi-normal if and only if it has the following properties:

(i) The localisation K_p is *Gorenstein* for any height-1 prime P.

(ii) Any ideal of grade 1 in K has height 1.

Now let the domain K be quasi-normal with a derivation d, and let $R = K[x, d]$. With a condition on d it is shown that a finitely generated left R-module M is reflexive if and only if there exists a finitely generated free left R-module F, containing M, such that any prime in K associated to F/M has height $\leqslant 1$. One other equivalent condition is that M is torsion-free and $\operatorname{Ext}^1_K(K/P, M) = 0$ for all primes P with height at least 2. We shall also investigate the necessity of conditions imposed on K, d and find properties for R corresponding to those of K.

Let R be any ring, M an R-module. The abelian group $\operatorname{Hom}_R(M, R)$, endowed with the usual R-module structure is denoted by M^*. The module M is called torsion-less if the natural homomorphism θ: $M \to M^{**}$ is a

Received by the editors March 21, 1978 and, in revised form, June 23, 1978.
AMS (MOS) subject classifications (1970). Primary 16A49; Secondary 13C10.
Key words and phrases. Reflexive module, ring of differential polynomials, quasi-normal.

monomorphism; M is called reflexive if θ is an isomorphism. Now suppose that R is a Noetherian integral domain with full ring of quotients Q. If M is a finitely generated torsion-free R-module then M is torsion-less. Let I be a nonzero finitely generated left R-submodule of Q. Then I^* is naturally isomorphic with $I^{-1} = \{q \in Q: Iq \subseteq R\}$. Clearly I is reflexive if and only if $I = (I^{-1})^{-1}$. Writing I^* for I^{-1}, we have that $I \subseteq I^{**}$ and $I^* = I^{***}$.

Again let K be a ring with a derivation d. An ideal I of K is called d-invariant if $d(I) \subseteq I$. If no nonzero proper ideal of K is d-invariant we say that K is d-simple. A useful formula giving the global dimension of $R = K[x, d]$ is due to K. R. Goodearl [2, Theorem 23] which we state as:

THEOREM 1.1. *Let $R = K[x, d]$ where K is commutative Noetherian of finite global dimension n. Set*

$$k = \sup\{pd_K(K/J): J \text{ is a primary } d\text{-invariant ideal}\}.$$

Then $\operatorname{gl\,dim} R = \max\{n, k + 1\}$.

2. Reflexivity for $R = K[x, d]$. *Throughout* this section K will be a commutative Noetherian domain (not a field) with a derivation d and $R = K[x, d]$. If k_1, k_2 is a K-sequence in K then R/Rk_1 is nonzero and k_2 is not a zero-divisor on R/Rk_1. For an ideal P of grade $\geqslant 2$ we thus, by the standard properties of grade, have that $\operatorname{Ext}_K^1(K/P, R) = 0$.

LEMMA 2.1. *Let k_1, k_2 be a K-sequence in K, M a left R-module such that $M^*k_1 \neq M^*$. Then k_2 is not a zero-divisor on M^*/M^*k_1.*

PROOF. The sequence $0 \to R \to R \xrightarrow{\rho} R/Rk_1 \to 0$, where ρ is right multiplication by k_1, is an exact sequence of left R-modules. This yields the exact sequence of right K-modules,

$$0 \to M^* \xrightarrow{\rho} M^* \to \operatorname{Hom}_R(M, R/Rk_1).$$

Thus M^*/M^*k_1 is isomorphic to a right K-submodule of $\operatorname{Hom}_R(M, R/Rk_1)$, hence it suffices to show that k_2 is not a zero-divisor on $\operatorname{Hom}_R(M, R/Rk_1)$. Let $f: M \to R/Rk_1$ be a nonzero R-homomorphism with $f(m) = r + Rk_1$ for some $m \in M$, $r \in R \setminus Rk_1$. If $fk_2 = 0$ then $0 = (fk_2)(m) = (f(m))k_2 = rk_2 + Rk_1$. This means that k_2 is a zero-divisor on R/Rk_1, which it is not.

Let P be any prime ideal of K. The derivation d can be uniquely extended to a derivation $\bar{d}: K_P \to K_P$. We now characterize reflexive R-modules when K is quasi-normal domain, K_P is $\bar{d}$-simple for all height-1 primes P. In the following, associated primes are always associated primes in K.

THEOREM 2.2. *Let K, d be as just stated. The following conditions are equivalent for a finitely generated left R-module M.*

(1) *M is reflexive as an R-module.*

(2) *M is a torsion-free R-module and* $\operatorname{Ext}_K^1(K/P, M) = 0$ *for all prime ideals P with* height $(P) \geqslant 2$.

(3) *There exists a finitely generated free left R-module F containing M such that any prime associated to F/M has height* $\leqslant 1$.

(4) *M is torsion-free as an R-module and whenever k_1, k_2 is a K-sequence in K with $k_1 M \neq M$ then k_2 is not a zero-divisor on M/k_1M.*

PROOF. (2) $\Rightarrow$ (1). Since M is a finitely generated torsion-free R-module and since R has a two-sided quotient division ring we can deduce that M is torsion-less. Let now P be a prime of height $\leqslant 1$ in K. The ring $S = K_P[x, \bar{d}]$ is a right and left partial quotient ring of R which is hereditary by Theorem 1.1. Thus tensoring the exact sequence

$$0 \to M \xrightarrow{\theta} M^{**} \to \operatorname{Coker} \theta \to 0,$$

with S it follows that $S \otimes_R N = 0$, where $N = \operatorname{Coker} \theta$. Suppose $N \neq 0$; then from $0 = S \otimes_R N \simeq (K_P \otimes_K R) \otimes_R N \simeq K_P \otimes_K N$ we deduce that any associated prime of N must have height $\geqslant 2$. Let P be an associated prime of N. Then we have an exact sequence

$$\operatorname{Hom}_K(K/P, M^{**}) \to \operatorname{Hom}_K(K/P, N) \to \operatorname{Ext}^1_K(K/P, M).$$

The first term of the above sequence is clearly zero, and the last term vanishes by (2). This contradiction shows that $N = 0$ and hence M is reflexive.

(3) $\Rightarrow$ (2) Let $0 \to M \to F \to F/M \to 0$ be exact with F the given finitely generated free. Clearly M is torsion-free. Let P be a prime in K with height $(P) \geqslant 2$. Then there is an exact sequence

$$\operatorname{Hom}(K/P, F/M) \to \operatorname{Ext}^1(K/P, M) \to \operatorname{Ext}^1(K/P, F).$$

Now $\operatorname{Hom}(K/P, F/M) = 0$ by (3); and since F is finitely generated there is an isomorphism $\operatorname{Ext}^1(K/P, F) \simeq \bigoplus \operatorname{Ext}^1(K/P, R)$. But grade $(P) \geqslant 2$, so that $\operatorname{Ext}^1(K/P, R) = 0$. Therefore $\operatorname{Ext}^1(K/P, M) = 0$.

(1) $\Rightarrow$ (4) M is torsion-free since by assumption $M \simeq (M^*)^*$. Now applying Lemma 2.1 (in its right-hand version) we deduce (4).

(4) $\Rightarrow$ (2) Assume that P is a prime in K with height $(P) \geqslant 2$. Then there is a K-sequence k_1, k_2 in P. Since by assumption M is torsion-free, k_1 is not a zero-divisor on M, and hence by [**6**, Theorem, p. 101],

$$\operatorname{Ext}^1_K(K/P, M) \simeq \operatorname{Hom}_K(K/P, M/k_1M).$$

If now $M/k_1M \neq 0$ it follows, by (4), that $\operatorname{Hom}_K(K/P, M/k_1M) = 0$ since $k_2 \in P$. Therefore $\operatorname{Ext}^1_K(K/P, M) = 0$.

That (1) implies (3) is easy.

Theorem 2.2 is a generalisation of [**4**, Theorem 3.7] which proved (when the base ring K is d-simple) that a nonzero proper left ideal I of R is reflexive if and only if any associated prime of R/I has height $\leqslant 1$.

In order to investigate the necessity of conditions imposed on K in the above theorem we first prove two lemmas.

LEMMA 2.3. *Grade and height* 1 *coincide in* K *if and only if any associated prime of* R/Rc, *where* c *is a nonzero nonunit in* R, *has height* $\leqslant 1$.

PROOF. Assume that grade and height 1 coincide in K and let P be a prime in K associated to R/Rc, where c is a nonzero nonunit in R. We must show that if $P \neq 0$, then height $(P) = 1$. Assume then $P \neq 0$, and let $r \in R \setminus Rc$ be such that $Pr \subseteq Rc$. So $RPr \subseteq Rc$ which gives $c^{-1}R \subseteq r^{-1}(RP)^*$. It is easy to see that $(RP)^* = P^*R$, and hence $RP^{**}r \subseteq Rc$ which then shows that P cannot have grade $\geqslant 2$. Therefore height $(P) = 1$.

Conversely assume that associated primes of R/Rc are all of height $\leqslant 1$. In order to show that grade and height 1 coincide in K it is sufficient to prove that any nonzero principal ideal of K is height unmixed. Let P be an associated prime of Kc, where c is a nonzero nonunit in K. There exists an ideal M containing c such that $M/Kc \simeq K/P$. Since R is K-free $R/RP \simeq R \otimes K/P \simeq R \otimes M/Kc \simeq RM/Rc$, so that R/RP is isomorphic to a submodule of R/Rc. Clearly $P \in \operatorname{Ass}(R/RP) \subseteq \operatorname{Ass}(R/Rc)$, hence by assumption height $(P) = 1$.

LEMMA 2.4. *Let* M *be a finitely generated left* R-*module such that the grade of any prime associated to* M *is at least* 2. *Then* $\operatorname{Ext}^1_R(M, R) = 0$.

PROOF. By assumption $0 \notin \operatorname{Ass}(M)$ and so 0 is not associated to any factor module of M. Now the proof of [**1**, Lemma 4] shows that there is a chain

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0,$$

of R-submodules such that $M_i/M_{i+1} \simeq R/RP_i$ $(i = 0, \ldots, n-1)$ where P_i is a nonzero prime in K. An induction argument on n using the fact that for a prime ideal P of K, $\operatorname{Ass}(R/RP) = \{P\}$ shows that

$$\operatorname{Ass}(M) \subseteq \{P_0, \ldots, P_{n-1}\} \subseteq \operatorname{Supp}(M)$$

and that these sets have the same minimal elements. Therefore each P_i is of grade at least 2. Since R is an integral domain $M^*_{n-1} = 0$ and hence $M_i^* = 0$, for all i. We now proceed by induction on n. If $n = 1$, then $\operatorname{Ext}^1_R(M, R) \simeq \operatorname{Ext}^1_R(R/RP_0, R) \simeq (RP_0)^*/R = 0$ since grade $(P_0) \geqslant 2$ implies that $P_0^* = K$ and so $(RP_0)^* = R$. Let $n > 1$, and assume the result for all finitely generated left R-modules which have series of length $< n$. Then from $0 \to M_1 \to M \to M/M_1 \to 0$ we obtain the exact sequence of right R-modules

$$0 \to \operatorname{Ext}^1_R(M/M_1, R) \to \operatorname{Ext}^1_R(M, R) \to \operatorname{Ext}^1_R(M_1, R).$$

The lemma now follows by induction.

THEOREM 2.5. *The following statements are equivalent.*

(1) K *is quasi-normal and* K_P *is* $\bar{d}$-*simple for all primes* P *in* K *of height* $\leqslant 1$.

(2) R *satisfies*: (i) *For nonzero left ideals* $I \subseteq J$ *of* R, $I^* = J^*$ *implies that* $\operatorname{Ext}^1_R(J/I, R) = 0$;

(ii) *A nonzero proper left ideal* I *of* R *is reflexive if and only if any associated prime of* R/I *has height at most* 1.

PROOF. (1) $\Rightarrow$ (2). We only need to show that (i) holds. Let $0 \neq I \subseteq J$ be left ideals in R with $I^* = J^*$. Since R has a hereditary two-sided partial quotient ring, namely $F[x, \bar{d}]$ where F is the quotient field of K, by **[5**, Lemma 2.1], the K-module J/I is torsion. Hence $0 \notin \mathrm{Ass}(J/I)$. Now let P be a height-1 prime associated to J/I. Then there is a left ideal L such that $I \subseteq L \subseteq J$ and $L/I \simeq R/RP$. Clearly $L^* = I^*$ and again by **[5**, Lemma 2.1], $R_P L = R_P I$ where $R_P = K_P[x, \bar{d}]$ which is hereditary since K_P is $\bar{d}$-simple. But then $R_P = R_P P$, an obvious contradiction. It follows that any associated prime of J/I has height $\geqslant 2$ and thus (i) follows by Lemma 2.4.

(2) $\Rightarrow$ (1). By Lemma 2.3 grade and height 1 coincide in K. Let P be a height-1 prime. To show that K_P is Gorenstein, by **[6**, Theorem 222], it is enough to prove that any nonzero ideal of K_P is reflexive. Let $I \subseteq P$ be a nonzero ideal of K such that I_P is nonreflexive. Thus $I \neq I^{**}$ and $P \in \mathrm{Supp}(I^{**}/I)$. Since P is a minimal element of $\mathrm{Supp}(I^{**}/I)$, there are ideals $I \subseteq L \subseteq M \subseteq I^{**}$ such that $M/L \simeq K/P$. Since $I^* = I^{***}$, we have $L^* = M^*$ and hence $(RL)^* = (RM)^*$ which by (i) implies that $\mathrm{Ext}_R^1(RM/RL, R) = 0$. But $RM/RL \simeq R/RP$, so that $0 = \mathrm{Ext}_R^1(R/RP, R) \simeq (RP)^*/R$. This gives grade $(P) \geqslant 2$ which is impossible. It remains to show that K_P is $\bar{d}$-simple. To see this we show that $S = K_P[x, \bar{d}]$ is hereditary and then apply Theorem 1.1. Any left ideal of S is of the form SI where I is a left ideal of R. Suppose that SI is a nonzero nonreflexive left ideal of S. Then the set of primes in K_P associated to $(SI)^{**}/SI$ is nonempty and contains only PK_P as K_P is a local domain of dimension 1 and $(SI)^{**}/SI$ is a nonzero torsion K_P-module. We can now deduce that $PK_P \in \mathrm{Ass}(SI^{**}/SI)$, and hence $P \in \mathrm{Ass}(I^{**}/I)$ since K is Noetherian. Now as before we can derive the contradiction grade $(P) \geqslant 2$. Thus in the Noetherian integral domain S (which is of finite global dimension) every nonzero left ideal is reflexive. It follows that S is hereditary and the proof is complete.

Theorem 2.5 should perhaps be compared with **[3**, Theorem 3.8] which proves that a commutative Noetherian domain K is quasi-normal if and only if K has the following properties:

(i) If $I \subseteq J$ are nonzero ideals in K, then $I^* = J^*$ implies $\mathrm{Ext}^1(J/I, K) = 0$.

(ii) Any nonzero reflexive ideal in K is unmixed of height one.

Finally using the methods of the proof for the above theorem and applying Lemma 2.4, the following corollary may easily be proved.

COROLLARY 2.6. *Let K be quasi-normal such that K_P is $\bar{d}$-simple whenever P is a height-*1 *prime. Let* $0 \neq I \subseteq J$ *be left ideals in R. Then*:

(i) *If* $0 \notin \mathrm{Ass}(J/I)$ *then* $I^* \neq J^*$ *if and only if* $\mathrm{Ass}(J/I)$ *contains a height-*1 *prime.*

(ii) $I^* = J^*$ *if and only if* $\mathrm{Ext}_R^1(J/I, R) = 0$.

(iii) *If* $\mathrm{Ass}(J/I) = \{P\}$ *and if J is reflexive then*

$$J^* \neq I^* \Leftrightarrow \text{height}\,(P) = 1 \Leftrightarrow I \textit{ is reflexive}.$$

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