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INNER FUNCTIONS ON THE POLYDISC[1]

S. H. KON

ABSTRACT. It is shown that the inner functions on the polydisc, unlike the classical case of the unit disc, fail to separate the points of the maximal ideal space of $H^\infty(U^n)$. From this we deduce that the inner functions generate a proper closed subalgebra of $H^\infty(U^n)$.

1. Introduction. Let T^n be the distinguished boundary of the unit polydisc U^n. The class of all bounded analytic functions on U^n is denoted by $H^\infty(U^n)$. It is well known that for $f \in H^\infty(U^n)$, the radial limit

$$f^*(\omega) = \lim_{r\to 1} f(r\omega)$$

exists for almost every $\omega \in T^n$ with respect to the normalised Lebesgue measure. A bounded analytic function is said to be an inner function if its radial boundary function is of absolute value one almost everywhere on T^n.

Let $J(U^n)$ denote the closed subalgebra generated by the inner functions in $H^\infty(U^n)$. Identifying a bounded analytic function with its radial boundary function f^*, $H^\infty(U^n)$ and $J(U^n)$ can be identified with closed subalgebras of $L^\infty(T^n)$.

With pointwise addition and multiplication, $H^\infty(U^n)$ and $L^\infty(T^n)$ are Banach algebras under the supremum norm. Let M_n and X_n be the maximal ideal space of $H^\infty(U^n)$ and $L^\infty(T^n)$ respectively. Define $\tau: X_n \to M_n$ by mapping each complex homomorphism of $L^\infty(T^n)$ to its restriction on $H^\infty(U^n)$. The Shilov boundary of $H^\infty(U^n)$ will be denoted by ∂_n and the Gelfand transform of an element f by $\hat{f}$. For the unit disc U, the index $n = 1$ will be omitted from our notations.

For the classical case of $H^\infty(U^n)$, it is well known that τ is a homeomorphism from X into M and $\partial = \tau X$, see [2]. For $n > 1$, $\tau: X_n \to M_n$ is still continuous, but as shown in [5], it is not one-to-one and $\partial_n \neq \tau X_n$.

A result of Hoffman together with a result of Douglas and Rudin show that the inner functions separate the points of all of the maximal ideal space M of $H^\infty(U)$, see [1].

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Recently, Marshall [4] has shown that the inner functions generate $H^\infty(U)$. From this it also follows that the inner functions separate the points of M.

Naturally the following two questions also arises for the polydisc.

(i) Do the inner functions separate the points of the maximal ideal space M_n for $n > 1$?

(ii) Do the inner functions generate $H^\infty(U^n)$ for $n > 1$?

We will answer both questions in the negative in this paper.

2. Separating points of the maximal ideal space. By considering inner functions of the form $(z_i - \alpha_i)/(1 - \bar{\alpha}_i z_i)$, for $\alpha_i \in U$, it is easy to see that the inner functions separate points of M_n from different fibers. Using an argument of Douglas and Rudin [1], and the fact that $\partial_n \neq \tau X_n$ for $n > 1$, it will be shown that the inner functions cannot separate the points of M_n. More precisely, we have the following theorem.

THEOREM 1. *The Gelfand transforms of the inner functions do not separate the points of the maximal ideal space M_n of $H^\infty(U^n)$ for $n > 1$.*

PROOF. Suppose that the Gelfand transforms of the inner functions separate the points of τX_n. Let E be a proper compact subset of τX_n, and let $\Phi_1 \in \tau X_n - E$. Now for each $\Phi_\alpha \in E$, there exists an inner function u_α such that

$$\hat{u}_\alpha(\Phi_1) \neq \hat{u}_\alpha(\Phi_\alpha)$$

by our assumption. Observe that $u \in H^\infty(U^n)$ is inner if and only if $|\hat{u}(\Phi)| = 1$ for every $\Phi \in \tau X_n$, see [3]. Hence without loss of generality, it can be assumed that $\hat{u}_\alpha(\Phi_1) = 1$ and $\operatorname{Re} \hat{u}_\alpha(\Phi_\alpha) < 1$. There is then, a neighbourhood $N(\Phi_\alpha)$ of Φ_α such that

$$\operatorname{Re} \hat{u}_\alpha(\Phi) < 1 \quad \text{for all } \Phi \in N(\Phi_\alpha).$$

By the compactness of E, there exists finitely many inner functions, say $u_1, \ldots, u_m$ such that

$$\hat{u}_k(\Phi_1) = 1 \quad \text{for } 1 \leqslant k \leqslant m$$

and

$$\inf_k \operatorname{Re} \hat{u}_k(\Phi) < 1 \quad \text{for every } \Phi \in E.$$

Then $f = 1 + u_1 + \cdots + u_m \in H^\infty(U^n)$ and $\hat{f}(\Phi_1) = m + 1 = \|\hat{f}\|$, but $|\hat{f}(\Phi)| < m + 1$ for every $\Phi \in E$. Thus E cannot contain ∂_n and we deduce that $\partial_n = \tau X_n$, contradicting the result of [5]! This shows that the inner functions cannot separate the points of τX_n and hence those of M_n.

Now if we assumed $J(U^n) = H^\infty(U^n)$, then the inner functions being generators of the commutative Banach algebra $H^\infty(U^n)$ must obviously separate points of the maximal ideal space M_n. For $n > 1$, this contradicts Theorem 1 and we have the following result.

THEOREM 2. *Let $J(U^n)$ be the closed subalgebra generated by the inner functions in $H^\infty(U^n)$. Then*

$$J(U^n) \neq H^\infty(U^n) \quad \text{for } n > 1.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MALAYA, KUALA LUMPUR 22-11, MALAYSIA