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A DILATION THEOREM FOR $|\cdot|$-PRESERVING MAPS OF $\mathcal{C}^*$-ALGEBRAS[1]

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ABSTRACT. A linear map of a (unital) $\mathcal{C}^*$-algebra into a $\mathcal{C}^*$-algebra, which preserves the absolute value, is analyzed as the composition of a (unital) $*$-homomorphism into a usually larger target algebra, followed by a suitable multiplication $x \mapsto xb = b^{1/2}xb^{1/2}$.

Two familiar classes of linear maps of $\mathcal{C}^*$-algebras preserve the absolute value: The first and more familiar is the $*$-homomorphism. The second is the map $x \mapsto xb = b^{1/2}xb^{1/2}$ of a sub-$\mathcal{C}^*$-algebra $\mathfrak{A}$ of $\mathfrak{B}$ into $\mathfrak{B}$, where b is a positive element of $\mathfrak{B}$ commuting with each element of $\mathfrak{A}$. It is not hard to see that the second class subsumes the first, and corresponds precisely to the case $b^2 = b$. Here we show that the second class is in fact all-inclusive (Corollary 3); but there is no suggestion of uniqueness in the construction.

In Theorem 2 we give a more canonical realization of such a map, by embedding the codomain algebra $\mathfrak{B}$ in $\mathfrak{B}^{dd}$, its "double dual" algebra. Here the analysis is essentially unique. The principal tool is the Kaplansky density theorem, adapted to the $*$-strong topology.

In a sequel, we will give necessary and sufficient conditions that a linear map of $\mathcal{C}^*$-algebras preserve the absolute value. The main result is that a linear map preserves absolute value if and only if it is 2-positive and preserves zero products (i.e., $xy = 0$ implies $\phi(x)\phi(y) = 0$).

THEOREM 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{C}^*$-algebras, $\mathfrak{A}$ unital, and $\phi: \mathfrak{A} \to \mathfrak{B}$ a linear map satisfying $\phi(|x|) = |\phi(x)|$, for all $x \in \mathfrak{A}$. Then*

(i) *$\phi(I)$ commutes with $\phi(x)$ for all $x \in \mathfrak{A}$;*

(ii) *there exist a unital $\mathcal{C}^*$-algebra $\mathfrak{D}$ containing $\mathfrak{B}$, and a unital $*$-homomorphism $\psi: \mathfrak{A} \to \mathfrak{D}$ such that $\phi(x) = \psi(x)\phi(I)$ $(x \in \mathfrak{A})$.*

PROOF. (i) ϕ is positive, so selfadjoint. If $u \in \mathfrak{A}$ is unitary, $|\phi(u)| = \phi(|u|) = \phi(I)$, so $|\phi(u^*)| = |\phi(u)^*| = |\phi(u)|$. It follows that $\phi(u)$ is normal, so commutes with $|\phi(u)| = \phi(I)$, for all unitary u in $\mathfrak{A}$. Since the unitary group of $\mathfrak{A}$ spans $\mathfrak{A}$, $\phi(I)$ commutes with $\phi(x)$ for all $x \in \mathfrak{A}$.

(ii) (α) We first prove a weakened version of condition (ii), in which $\mathfrak{D}$ (resp. ψ) is not required to be unital. Let $(\pi, \mathcal{H})$ be a faithful $*$-representation

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of $\mathfrak{B}$, and put $b = \phi(I)$. Then $\pi(b) \geqslant 0$, with null space $n(\pi(b))$, and support $s(\pi(b)) = n(\pi(b))^{\perp}$.

Define (possibly unbounded) selfadjoint T on $\mathcal{H}$ by

$$T = \begin{cases} \pi(b)^{-1} & \text{on } s(\pi(b)), \\ 0 & \text{on } n(\pi(b)). \end{cases}$$

Since ϕ and π are positive, if $0 \leqslant x \leqslant I$ in $\mathfrak{A}$, $0 \leqslant \pi(\phi(x)) \leqslant \pi(b)$. From this and the fact that T is affiliated with the commutant of $\pi \circ \phi(\mathfrak{A})$ it follows that $0 \leqslant T\pi(\phi(x)) = \pi(\phi(x))T \leqslant I$ on $\mathfrak{D}_T$. Then for all $x \in \mathfrak{A}$, $T\pi(\phi(x)) = \pi(\phi(x))\, T$ extends to a bounded operator $\lambda(x)$ on $\mathcal{H}$.

Let $\mathfrak{S}$ be the $\mathcal{C}^*$-algebra generated in $\mathcal{L}(\mathcal{H})$ by $\lambda(\mathfrak{A})$ and $\pi(\mathfrak{B})$. The map λ: $\mathfrak{A} \to \mathfrak{S}$ defined above is linear and as we will show, preserves absolute values: In fact, with $0 \leqslant x \leqslant I$ in $\mathfrak{A}$, and $S = \pi(\phi(x))$, $S \leqslant \pi(b)$ implies $S^2 \leqslant \pi(b)^2$, since $\pi(b)$ and S commute; so $S^2T^2 = T^2S^2 \leqslant I$. Now if $x \in \mathfrak{A}$ and $\|x\| \leqslant 1$, put $R = \pi(\phi(x))$. Then $|R| = \pi \circ \phi(|x|) = S$ of the type just studied, so $I \geqslant S^2T^2 = R^*RT^2 = TR^*RT = |RT|^2 = (ST)^2$, and $ST \geqslant 0$, so $|RT| = ST = |R|T$, or $|\lambda(x)| = \lambda(|x|)$, as desired.

If we identify $\mathfrak{B}$ with its faithful image $\pi(\mathfrak{B})$ in $\mathfrak{S}$, we have (since $s[\phi(x)] = s[|\phi(x)|] = s[\phi(|x|)] \leqslant s[\phi(I)] = T\phi(I) = \lambda(I)$) λ: $\mathfrak{A} \to \lambda(I)\mathfrak{S}\lambda(I)$ preserves I and absolute values, so by [4, Corollary 4], λ is a $*$-homomorphism. But $s[\phi(x)] \leqslant T\phi(I)$ also provides that $\phi(x) = \lambda(x)\phi(I)$.

Thus every linear ϕ: $\mathfrak{A} \to \mathfrak{B}$ which preserves absolute values has a $*$-homomorphic dilation $(\lambda, \mathfrak{S})$.

(β) We treat next the special, tractable case in which the given map ϕ is a (nonunital) $*$-homomorphism.

Adapting our notation to the foregoing, we assume $\mathfrak{A}$ unital, and λ: $\mathfrak{A} \to \mathfrak{S}$ $*$-homomorphic, and we seek a unital $*$-homomorphic dilation $(\psi, \mathfrak{D})$ of $(\lambda, \mathfrak{S})$: that is, a unital $\mathcal{C}^*$-algebra $\mathfrak{D}$ containing $\mathfrak{S}$, and a unital $*$-homomorphism ψ: $\mathfrak{A} \to \mathfrak{D}$ satisfying

$$\lambda(x) = \psi(x)\lambda(I).$$

In fact, we will construct for each essential $*$-representation $(\sigma, \mathcal{K})$ of $\mathfrak{A}$ a unital $*$-homomorphic dilation $(\psi_\sigma, \mathfrak{D}_\sigma)$ of $(\lambda, \mathfrak{S})$, as follows:

Let $(\rho, \mathcal{J})$ be a faithful, essential $*$-representation of $\mathfrak{S}$, and let $(\tilde{\rho}, \mathcal{H}) = (\rho(\cdot) \otimes I_{\mathcal{K}}, \mathcal{J} \otimes \mathcal{K})$. Put $P = \rho \circ \lambda(I)$, $Q = I_{\mathcal{J}} - P$ in $\mathcal{L}(\mathcal{J})$, and write $\mathcal{H} = P\mathcal{J} \otimes \mathcal{K} \oplus Q\mathcal{J} \otimes \mathcal{K}$. The first summand is $P \otimes I_{\mathcal{K}}\mathcal{H} = \tilde{\rho} \circ \lambda(I)\mathcal{H}$, the essential space for the representation $\tilde{\rho} \circ \lambda$ of $\mathfrak{A}$ on $\mathcal{H}$.

Now put $\mathfrak{D}_\sigma = \mathcal{L}(\mathcal{H})$, and identify $\mathfrak{S}$ with its faithful image $\tilde{\rho}(\mathfrak{S}) \subset \mathfrak{D}_\sigma$. If $\psi_\sigma = \lambda \oplus I_{Q\mathcal{J}} \otimes \sigma(\cdot)$: $\mathfrak{A} \to \mathfrak{D}_\sigma$, then $(\psi_\sigma, \mathfrak{D}_\sigma)$ is a unital, $*$-homomorphic dilation of $(\lambda, \mathfrak{S})$.

(γ) Finally, we compose the results of (α) and (β). If $(\phi, \mathfrak{B})$ is as in the hypothesis of the theorem, (α) yields the $*$-homomorphic dilation $(\lambda, \mathfrak{S})$; let $(\psi_\sigma, \mathfrak{D}_\sigma)$ be a unital dilation of $(\lambda, \mathfrak{S})$ given by (β). Then $\mathfrak{B} \subset \mathfrak{S} \subset \mathfrak{D}_\sigma$, ψ_σ: $\mathfrak{A} \to \mathfrak{D}_\sigma$ is a unital $*$-homomorphism, and for $x \in \mathfrak{A}$,

$$\phi(x) = \lambda(x)\phi(I) = \psi_\sigma(x)\lambda(I)\phi(I) = \psi_\sigma(x)\phi(I),$$

so $(\psi_\sigma, \mathfrak{D}_\sigma)$ is a unital *-homomorphic dilation of $(\phi, \mathfrak{B})$. The proof is complete. □

COROLLARY 1. *With* $\mathfrak{A}$, $\mathfrak{B}$, ϕ *as in the theorem,* $\phi^{-1}(0)$ *is a closed, two-sided ideal in* $\mathfrak{A}$.

PROOF. Take λ as constructed in part (α) of the proof of the theorem. Then $\phi^{-1}(0) = \ker \lambda$. □

REMARK. Corollary 1 also has a simple, direct proof, valid with no unital condition on $\mathfrak{A}$.

COROLLARY 2. *Let* $\mathfrak{A}$, $\mathfrak{B}$, ϕ *be as in the statement of the theorem, and let* J *be any closed, two-sided ideal in* $\mathfrak{A}$. *Then there exist a* $\mathcal{C}^*$*-algebra* $\mathfrak{D}_J$ *containing* $\mathfrak{B}$ *and a unital* *-*homomorphism* $\psi_J: \mathfrak{A} \to \mathfrak{D}_J$ *satisfying*

(a) $\ker \psi_J = J \cap \phi^{-1}(0)$,

(b) $\phi(x) = \psi_J(x)\phi(I)$ $(x \in \mathfrak{A})$.

PROOF. In the notation of the proof of the theorem,

$$\ker \psi_\sigma = \ker \lambda \cap \ker \sigma = \phi^{-1}(0) \cap \ker \sigma.$$

We need only choose σ so that $\ker \sigma = J$. □

Especially, with $J = \{0\}$, we have

COROLLARY 3. *If* $\mathfrak{A}$ *is a (unital)* $\mathcal{C}^*$*-algebra,* $\mathfrak{B}$ *a* $\mathcal{C}^*$*-algebra, and* ϕ: $\mathfrak{A} \to \mathfrak{B}$ *a linear map satisfying* $\phi(|x|) = |\phi(x)|$ $(x \in \mathfrak{A})$, *then there exists a unital* $\mathcal{C}^*$*-algebra* R *such that*

(i) R *contains* $\mathfrak{A}$ *as a (unital)* $\mathcal{C}^*$*-subalgebra,*

(ii) R *contains* $\mathfrak{B}$ *as a* $\mathcal{C}^*$*-subalgebra,*

(iii) ϕ *is the restriction to (embedded)* $\mathfrak{A}$ *of the map* $x \mapsto x\phi(I)$ *of* R *into* R.

One could of course replace condition (iii) *by*

(iii)′ ϕ *is the restriction to* $\mathfrak{A}$ *of the completely positive map* $x \mapsto \phi(I)^{1/2} x \phi(I)^{1/2}$ *of* R *into* R.

Here, if $\mathfrak{A}$ *is not unital,* $\phi(I)$ *is to be interpreted as the image in the second adjoint ("double dual") algebra* $(\mathfrak{B}^{\mathrm{d}})^{\mathrm{d}} = \mathfrak{B}^{\mathrm{dd}}$ *of* $I \in \mathfrak{A}^{\mathrm{dd}}$ *under the double transpose* $\phi^{\mathrm{dd}}: \mathfrak{A}^{\mathrm{dd}} \to \mathfrak{B}^{\mathrm{dd}}$.

The proof of Corollary 3 for the unital case is subsumed in the foregoing. The reduction of the nonunital case to the unital will be accomplished in the following lemma, and is further illumined in Theorem 2.

LEMMA 1. *Suppose that* $\phi: \mathfrak{A} \to \mathfrak{B}$ *is a linear map of* $\mathcal{C}^*$*-algebras satisfying* $\phi(|a|) = |\phi(a)|$ $(a \in \mathfrak{A})$. *Then* $\phi^{\mathrm{dd}}: \mathfrak{A}^{\mathrm{dd}} \to \mathfrak{B}^{\mathrm{dd}}$ *satisfies the same identity for* $a \in \mathfrak{A}^{\mathrm{dd}}$.

PROOF. We use the variant of Kaplansky's density theorem **[2]** obtained by replacing the strong topology by the *-strong topology, in both conclusion and proof. (Recall that the *-strong topology on $\mathcal{L}(\mathcal{H})$ is the weakest locally convex topology stronger than the strong operator topology in which the map $T \to T^*$ is continuous. A family of defining seminorms is $\{T \mapsto \|T\xi\|: \xi \in \mathcal{H}\} \cup \{T \mapsto \|T^*\xi\|: \xi \in \mathcal{H}\}$.) In the version of the proof in Dixmier **[1]**, for instance, one would append to the proof as it appears the observation that also $R_{12} = R_{21}^*$ tends strongly to S^*. Since the two topologies coincide on the space of selfadjoint elements, this secures the result. Thus the unit ball of $\mathfrak{A}$ is *-strongly dense in that of $\mathfrak{A}^{dd}$. Here $\mathfrak{A}$ is viewed as canonically embedded in $\mathfrak{A}^{dd}$, and both may be assumed to act on the universal representation space of $\mathfrak{A}$ [3].

Now if $a \in \mathfrak{A}^{dd}$, and $\|a\| \leq 1$, let $(a_\nu)_{\nu \in D}$ be a net in $\mathfrak{A}$ with $\|a_\nu\| \leq 1$, converging *-strongly to a. Then $a_\nu^* a_\nu \to a^*a$ (*-)strongly, so $|a_\nu| \to |a|$ strongly. Since, ϕ^{dd}: $\mathfrak{A}^{dd} \to \mathfrak{B}^{dd}$ is strongly and *-strongly continuous, (see Appendix) we have

$$\phi^{dd}(|a|) = \text{st.} \lim_\nu \phi(|a_\nu|) = \text{st.} \lim_\nu |\phi(a_\nu)| = |\phi^{dd}(a)|.$$

This proves the lemma, and so also the corollary. □

In the next theorem, we give a description of the mapping ϕ in more canonical terms.

THEOREM 2. *Let* ϕ: $\mathfrak{A} \to \mathfrak{B}$ *be a linear mapping of* $\mathcal{C}^*$*-algebras satisfying* $\phi(|a|) = |\phi(a)|$, *for all* $a \in \mathfrak{A}$. *Then there are a unique positive element* $b \in \mathfrak{B}^{dd}$ *supported on* $\bigcup_{a \in \mathfrak{A}}$ range $\phi(a)$, *and a unique* *-*homomorphism* ψ: $\mathfrak{A} \to \mathfrak{B}^{dd}$, *satisfying* $\phi(a) = b\psi(a)$, *for all* $a \in \mathfrak{A}$.

PROOF. *Existence*: We can apply the lemma, restrict ϕ^{dd} to $\mathfrak{A} + \mathbf{C}I$ ($= \mathfrak{A}$, if $\mathfrak{A}$ has a unit), and employ the arguments of parts (i) and (ii) (α) of the proof of Theorem 1 with (π, $\mathcal{H}$) the universal representation of $\mathfrak{B}$ extended as usual to $\mathfrak{B}^{dd}$. With $b = \phi^{dd}(I)$, and the observation that $b^{-1}\phi(a) =$ st. $\lim_n (I/n + b)^{-1}\phi(a) \in \mathfrak{B}^{dd}$, the proof of existence is complete.

Uniqueness: If ψ: $\mathfrak{A} \to \mathfrak{B}^{dd}$ is a *-homomorphism, $b \in \mathfrak{B}^{dd}$ is supported on $\bigcup_{a \in \mathfrak{A}}$ range $\phi(a)$, and $\phi(a) = b\psi(a)$ for all $a \in \mathfrak{A}$, b is determined on

$$\text{range } \psi(a) = \text{supp } \psi(a^*) \supset \text{supp } \phi(a^*) = \text{range } \phi(a)$$

by this condition, for each $a \in \mathfrak{A}$. Therefore $b = \phi^{dd}(I)$. It then follows trivially that $\psi = b^{-1}\phi$, and the proof is complete. □

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Appendix

SUBLEMMA. ϕ^{dd}: $\mathfrak{A}^{dd} \to \mathfrak{B}^{dd}$ *is strongly (and* *-*strongly) continuous.*

PROOF. *-strong continuity follows from strong continuity and self-adjointness. ϕ is of course norm continuous, because the algebraic transpose ϕ^*: $\mathfrak{B}^* \to \mathfrak{A}^*$ maps positive linear functionals (p.l.f.'s) to p.l.f.'s, so $\mathfrak{B}^d$ to $\mathfrak{A}^d$. Normalizing, we may assume ϕ contractive. If ρ is a state of $\mathfrak{B}$, we want first to show that $\rho(\phi(x)^*\phi(x)) \leq \rho[\phi(x^*x)] = (\phi^d(\rho))(x^*x)$ for $x \in \mathfrak{A}$. This is the strong continuity of ϕ: $\mathfrak{A} \to \mathfrak{B}$. For this, it suffices of course to show that

$$\phi(x)^*\phi(x) = |\phi(x)|^2 = (\phi(|x|))^2 \leq \phi(|x|^2) = \phi(x^*x),$$

or $(\phi(a))^2 \leq \phi(a^2)$ for $a \geq 0$ in $\mathfrak{A}$. This depends only on the restriction of ϕ to the commutative $\mathcal{C}^*$-subalgebra generated by a–thus we may and do assume for this part of the proof that $\mathfrak{A}$ is commutative. Then since x is normal if and only if $|x| = |x^*|$, $\phi(\mathfrak{A})$ is a selfadjoint linear space consisting of normal elements of $\mathfrak{B}$, so that $\phi(\mathfrak{A})$ is contained in a commutative subalgebra of $\mathfrak{B}$: we may assume $\mathfrak{B}$ commutative. Then $\mathfrak{B}^{\mathrm{dd}}$ is commutative, and in particular $b = \phi^{\mathrm{dd}}(I)$ centralizes $\phi(\mathfrak{A})$. Define for $x \in \mathfrak{A}^{\mathrm{dd}}$, $\psi(x) = b^{-1}\phi^{\mathrm{dd}}(x)$. As we saw in the proof of Theorem 1, ψ is positive and unital on $\mathfrak{A}^{\mathrm{dd}}$ to supp $b\mathfrak{B}^{\mathrm{dd}}$ supp b. By Kadison's Schwarz inequality [**5**], $(\psi(|x|))^2 \leqslant \psi(|x|^2)$ for $x \in \mathfrak{A}^{\mathrm{dd}}$; especially, $(b^{-1}\phi(|x|))^2 \leqslant b^{-1}\phi(|x|^2)$ for $x \in \mathfrak{A}$, so $\phi(|x|)^2 \leqslant b\phi(|x|^2) \leqslant \phi(|x|^2)$, since $\|b\| \leqslant 1$. We have shown ϕ strongly continuous on $\mathfrak{A}$. So ϕ has a strongly continuous extension $\tilde{\phi}$: $\mathfrak{A}^{\mathrm{dd}} \to \mathfrak{B}^{\mathrm{dd}}$. Using the relations

$$\text{weak} = \sigma\text{-weak} \leqslant \sigma\text{-strong} = \text{strong}$$

among topologies, valid for $\mathfrak{A}^{\mathrm{dd}}$ (respectively $\mathfrak{B}^{\mathrm{dd}}$) in their universal representations, it is routine to show that $\tilde{\phi} = \phi^{\mathrm{dd}}$. The sublemma is proved. □

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