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INVARIANT SUBSPACES OF VON NEUMANN ALGEBRAS. II

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ABSTRACT. It is shown that every parareductive operator algebra $A \subset B(H)$ (as defined below) is a von Neumann algebra. For the proof of this result, some new properties of paraclosed operators are obtained. Finally, a sufficient condition that a reductive algebra be a von Neumann algebra is given.

In what follows, H denotes a complex Hilbert space. The algebra of all bounded operators on H is denoted by $B(H)$. If $A \subset B(H)$, A' denotes the commutant of A, and A'' the bicommutant. Also, we denote by M_A the von Neumann algebra generated by A. For every $n \in \mathbf{N}$, we denote $H^{(n)} = \bigoplus_{i=1}^{n} H_i$ where $H_i = H$ for every i, $1 \leqslant i \leqslant n$, and

$$A^{(n)} = \{\underbrace{a \oplus a \oplus \cdots \oplus a}_{n} | a \in A\}.$$

The collection of all closed linear subspaces of H invariant under A (i.e. invariant under every $a \in A$) is denoted by $\operatorname{Lat} A$. A weakly closed algebra $A \subset B(H)$ is reductive [8] if $1 \in A$, and $\operatorname{Lat} A = \operatorname{Lat} M_A$. A linear subspace $K \subset H$ is paraclosed [4] if there exist a Hilbert space H_0 and a bounded linear operator Q: $H_0 \mapsto H$ such that $QH_0 = K$. The collection of all paraclosed subspaces of H, invariant under A is denoted $\operatorname{Lat}_{1/2} A$. A weakly closed algebra $A \subset B(H)$ will be called parareductive if $1 \in A$, and $\operatorname{Lat}_{1/2} A = \operatorname{Lat}_{1/2} M_A$. In this paper (Theorem 2.1) we show that if A is a parareductive algebra then A is a von Neumann algebra. In order to prove this result, in §1, we give some new results on paraclosed operators.

Finally, in §3 we prove a result, announced (without proof) in [7].

Since the paper was written a proof of Theorem 2.1 in the separable case has appeared [1]. Our approach covering the possibly nonseparable case is entirely different and in fact simplifies Azoff's proof. I am indebted to the referee for calling my attention to Azoff's paper and for the suggestion that Theorem 2.1 can be formulated in this general form.

1. Paraclosed operators. Let H_1, H be Hilbert spaces. A linear transformation S: $\mathfrak{D}_S \mapsto H(\mathfrak{D}_S \subset H_1)$ is *paraclosed* [4] if its graph $\Gamma_S = \{\xi \oplus S\xi | \xi \in$

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$\mathfrak{D}_S\}$ is a paraclosed subspace of $H_1 \oplus H$.

A linear transformation S: $\mathfrak{D}_S \mapsto H(\mathfrak{D}_S \subset H_1)$ is *semiclosed* [3] if there exist a Hilbert space H_2 and two closed operators S_1: $\mathfrak{D}_S \mapsto H_2$, S_2: $\mathfrak{D}_{S_2} \mapsto H(\mathfrak{D}_{S_2} \subset H_2)$ such that $S = S_2 S_1$. The following proposition shows that these two notions are equivalent.

1.1. PROPOSITION. *Let H_1, H be Hilbert spaces, and let S: $\mathfrak{D}_S \mapsto H(\mathfrak{D}_S \subset H_1)$ be a linear transformation. The following are equivalent*:

(a) *S is paraclosed*,

(b) *S is semiclosed.*

PROOF. (a) $\Rightarrow$ (b). Since S is paraclosed, its graph Γ_S is a paraclosed subspace of $H_1 \oplus H$. By the definition of paraclosed subspaces, there exist a Hilbert space H_2 and a bounded linear operator Q: $H_2 \mapsto H_1 \oplus H$ such that $QH_2 = \Gamma_S$. We may suppose that Q is injective, since otherwise we replace H_2 by $(\ker Q)^{\perp}$. Then, for every $\xi \in \mathfrak{D}_S$ there exists a unique $S_1(\xi) \in H_2$ such that $QS_1(\xi) = \xi \oplus S(\xi)$. Obviously, S_1 is a linear transformation S_1: $\mathfrak{D}_S \mapsto H_2$. Moreover, S_1 is a closed operator with $\mathfrak{D}_{S_1} = \mathfrak{D}_S$. Indeed, let $\{\xi_n\}_{n=1}^{\infty} \subset \mathfrak{D}_S$ be a sequence such that $\lim_n \xi_n = \xi_0$ and $\lim_n S_1(\xi_n) = \eta_0$. Since Q is continuous, it follows that $\lim_n QS_1(\xi_n) = Q(\eta_0)$. On the other hand $QS_1(\xi_n) = \xi_n \oplus S(\xi_n)$ and hence $Q(\eta_0) = \xi_0 \oplus S\xi_0$. It follows that $\xi_0 \in \mathfrak{D}_S$, and $S_1(\xi_0) = \eta_0$. Therefore S_1 is closed and $\mathfrak{D}_{S_1} = \mathfrak{D}_S$. If p_H is the projection of $H_1 \oplus H$ onto H, then $S_2 = p_H Q$ is a bounded operator, and $S = S_2 S_1$, whence S is semiclosed.

(b) $\Rightarrow$ (a). Let H_2 be a Hilbert space and let S_1: $\mathfrak{D}_S \mapsto H_2$, S_2: $\mathfrak{D}_{S_2} \mapsto H(\mathfrak{D}_{S_2} \subset H_2)$ be closed operators such that $S = S_2 S_1$. Let $H_0 = H_1 \oplus H_2 \oplus H$, and $H_3 = \{\xi \oplus S_1\xi \oplus S\xi | \xi \in \mathfrak{D}_S\}$. If q is the projection of H_0 onto $H_1 \oplus \{0\} \oplus H$, we see that $qp_{H_3} H_0 = \Gamma_S$ and therefore S is paraclosed, which completes the proof of proposition. In [3] it is proved that if S and T are semiclosed operators, then so are $S + T$ and ST (whenever the latter are defined). Therefore

1.2. COROLLARY. *If S and T are paraclosed operators, then so are $S + T$ and ST (whenever the latter are defined). We need also the following*:

1.3. PROPOSITION [4]. *Let S: $\mathfrak{D}_S \mapsto H(\mathfrak{D}_S \subset H_1)$ be a paraclosed operator. If $\mathfrak{D}_S$ is closed, then S is continuous.*

2. Parareductive algebras.

2.1. THEOREM. *Let $A \subset B(H)$ be a parareductive algebra. Then $A = M_A$.*

For the proof of this theorem we need some preliminary results.

Recall that a von Neumann algebra N has property (P) if for every $x \in B(H)$ the weakly closed convex hull of $\{u^* x u | u \in N, \text{ unitary}\}$ has nonvoid intersection with the commutant N' of N [9]. It is known that every discrete von Neumann algebra has property (P) [9].

The next lemma is due to D. Voiculescu [**12**]. We include his proof for the convenience of the reader.

2.2. LEMMA. *Let Z be a commutative von Neumann algebra, and let A be a parareductive algebra such that $Z \subset A \subset Z'$ and $M_A = Z'$. Then $A' = Z$.*

PROOF. Since $Z \subset A \subset Z'$, it follows that $Z \subset A' \subset Z'$. So, to prove the lemma, we must show that $A' \subset Z$. Let $t \in A'$. Then, by [**10**, Proposition 6.4], there exists $z \in Z$, such that for every projection $p \in Z$, the element $(z - t)p$ has no inverse in the algebra $Z'p$. Obviously $\ker(z - t)$ Lat $A \subset \mathrm{Lat}_{1/2}A = \mathrm{Lat}_{1/2}Z'$ and therefore the projection p_0 onto $\ker(z - t)$ is in Z. The element $(z - t)(1 - p_0)$ of $Z'(1 - p_0)$ is injective. We show that it is equal to zero. Indeed, if this is not true, then $0 \neq \mathrm{Range}[(z - t)(1 - p_0)] \in \mathrm{Lat}_{1/2}Z'(1 - p_0)$. By [**11**, Théorème 2] it follows that there exists a positive $z_0 \in Z(1 - p_0)$ such that $\mathrm{Range}[(z - t)(1 - p_0)] = \mathrm{Range}\, z_0$. By the spectral theorem, it follows that there exists a spectral projection $0 \neq p \leqslant 1 - p_0$ of z_0 such that

$$pH \subset \mathrm{Range}\, z_0 = \mathrm{Range}[(z - t)(1 - p_0)].$$

Then $(z - t)p$ is invertible in $Z'p$ which is impossible. Therefore $(z - t)(1 - p_0) = 0$, and hence $t = z \in Z$.

2.3. LEMMA. *Let $A \subset B(H)$ be a parareductive algebra. Then $A' = M_A'$.*

PROOF. Since A is parareductive, it follows that A is reductive. Let Z be the center of M_A. According to [**5**, Corollary 1] we have $Z \subset A''$. Since $A \subset A'' \subset M_A$ and A is parareductive, A'' is parareductive too.

Let $p \in Z$ be the maximal projection such that $M_A p$ is abelian and of infinite uniform multiplicity. Then, by [**5**, Theorem 3] it follows that $[(1 - p)A]' = (1 - p)M_A'$. Therefore $(1 - p)A' = (1 - p)M_A'$.

By the preceding lemma, we have $(pA'')' = pM_A'$, so $pA' = pM_A'$. Hence $A' = M'$.

The following lemma is a consequence of [**7**, Corollary 1.3].

2.4. LEMMA. *Let $A \subset B(H)$ be a parareductive algebra, and $b \in M$. Suppose that every paraclosed, densely defined operator that commutes with A, commutes also with b. Then $b \in A$.*

2.5. PROPOSITION. *Let $A \subset B(H)$ be a parareductive algebra. If M_A has property* (P), *then $A = M_A$.*

PROOF. We shall verify the hypothesis of Lemma 2.4. By Lemma 2.3 we have $A' = M_A'$. Let S: $\mathfrak{D}_S \mapsto H(\mathfrak{D}_S \subset H)$ be a paraclosed densely defined operator, that commutes with A.

It is easy to see that $\mathfrak{D}_S \in \mathrm{Lat}_{1/2}A = \mathrm{Lat}_{1/2}M_A$. By [**11**, Théorème 2] there exists $m' \in M_A'$, $m' \geqslant 0$ such that $\mathfrak{D}_S = m'H$. Since s is paraclosed and m' continuous, by Corollary 1.2, it follows that Sm' is a paraclosed operator. Since $\mathfrak{D}_{Sm'} = H$, by Proposition 1.3 we have Sm' continuous. Since S commutes with A and $m' \in M_A' = A'$, it follows that $Sm' \in A' = M_A'$.

Now, we show that S commutes with M_A. Let $m \in M_A$. Then we have:

$$mSm' = Sm'm = Smm'.$$

Therefore $mS = Sm$ on $m'H = \mathfrak{D}_S$ and the proposition is proved.

2.6. COROLLARY. *Let $A \subset B(H)$ be a parareductive algebra. If M_A is a discrete von Neumann algebra, then $A = M_A$.*

PROOF. Since every discrete von Neumann algebra has property (P), the corollary follows immediately from Proposition 2.6.

The following lemma is a consequence of [**1**, Proposition 4.1].

2.7. LEMMA. *Let $A \subset B(H)$ be a parareductive algebra, and Z the center of M_A. If $p \in Z$ is the projection such that pM_Ap is discrete and $(1-p)M_A(1-p)$ is continuous, then $p \in A$.*

The proof of Theorem 2.1 follows from Corollary 2.6., [**7**, Corollary 1.3] or [**11**, Théorème 1], and Lemma 2.7.

2.8. COROLLARY. *Let $A \subset B(H)$ be a weakly closed algebra such that $1 \in A$. If every linear subspace of H invariant under A is invariant under M_A, then $A = M_A$.*

This corollary has been obtained independently by E. Azoff [**2**].

3. Reductive algebras. In [**7**, Theorem 2.2] the following is proved.

3.1. THEOREM. *Let $A \subset B(H)$ be an algebra with the following properties:*

(a) *$A^{(2)}$ is reductive,*

(b) *$A^{(2)}$ contains a von Neumann algebra N with the property* (P) *and having finite commutant.*

Then $A = M_A$.

At the end of the same paper, we claimed (without proof) that the following improvement of this theorem can be given (see the proof below):

3.2. THEOREM. *Let $A \subset B(H)$ be an algebra with the following properties:*

(a) *$A^{(2)}$ is reductive,*

(b) *A contains a von Neumann algebra N having finite commutant.*

Then $A = M_A$.

We recall that a linear transformation T: $\mathfrak{D}_T \mapsto H$, ($\mathfrak{D}_T \subset H$) is a graph transformation for A [**8**] if there exist $n \in \mathbf{N}$ and linear transformations $\{T_i\}_{i=1}^{n-2}$ defined on $\mathfrak{D}_T$ such that $\{\xi \oplus T\xi \oplus T_1\xi \oplus \cdots \oplus T_{n-2}\xi | \xi \in \mathfrak{D}_T\} \in \operatorname{Lat} A^{(n)}$.

To prove the Theorem 3.2 we need the following:

3.3. LEMMA. *Let $N \subset B(H)$ be a von Neumann algebra, having finite commutant. Then, every densely defined graph transformation for N, is preclosed.*

PROOF. Let T be a densely defined graph transformation of N and let $\{T_i\}_{i=1}^{n-2}$ be linear transformations defined on $\mathfrak{D}_T$ such that $\{\xi \oplus T\xi \oplus T_1\xi \oplus \cdots \oplus T_{n-2}\xi | \xi \in \mathfrak{D}_T\} \in \operatorname{Lat} N^{(n)}$. If Δ_{n-1} is the diagonal of $H^{(n-1)}$, then it is easy to see that the transformation $\tilde{T}$: $(\mathfrak{D}_T^{(n-1)} \cap \Delta_{n-1}) \oplus \Delta_{n-1}^{\perp} \mapsto H^{(n-1)}$ defined by:

$$\tilde{T}(\xi \oplus \cdots \oplus \xi) = T\xi \oplus T_1\xi \oplus \cdots \oplus T_{n-2}\xi \quad \text{if } \xi \in \mathfrak{D}_T,$$

$$\tilde{T}(\xi_1 \oplus \cdots \oplus \xi_{n-1}) = 0 \oplus 0 \oplus \cdots \oplus 0 \quad \text{if } \xi_1 \oplus \cdots \oplus \xi_{n-1} \in \Delta_{n-1}^{\perp}$$

is a densely defined, closed operator affiliated to $N^{(n-1)'}$. Since N' is finite, it follows that $N^{(n-1)'}$ is a finite von Neumann algebra. Let p be the projection of $H^{2(n-1)}$ onto its nth component. Then $p \in N^{(n-1)'}$. Since $\tilde{T}$ is affiliated to $N^{(n-1)'}$, then according to [6, Theorem XV, p. 119] we obtain that $p\tilde{T}$ is preclosed, whence T is preclosed.

The proof of Theorem 3.2, is similar with the proof of [7, Theorem 2.2] applying Lemma 3.3 instead of [7, Lemma 2.3].

3.4. COROLLARY. *Let N, $M \subset B(H)$ be two type* II *von Neumann algebras such that $N \subset M$ and N' is finite. Then every reductive algebra A with $N \subset A \subset M$ is a von Neumann algebra.*

PROOF. Since M is a type II von Neumann algebra, M' is also a type II von Neumann algebra. Therefore, there exist p_1, $p_2 \in M'$ orthogonal projections which are equivalent and $p_1 + p_2 = 1$. A standard argument shows that the algebras A and $(p_1Ap_1)^{(2)}$ are unitarily equivalent.

Since A is reductive, it follows that $(p_1Ap_1)^{(2)}$ is reductive. The von Neumann algebra p_1Np_1 has finite commutant because $p_1 \in M' \subset N'$. Therefore the algebra p_1Ap_1 satisfies the hypothesis of Theorem 3.2. By this theorem p_1Ap_1 is a von Neumann algebra and hence A is a von Neumann algebra.

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