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APPLICATIONS OF SHRINKABLE COVERS

J. C. SMITH

ABSTRACT. An open cover $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ of a topological space X is shrinkable if there exists a closed cover $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$.

In this paper the author determines conditions necessary for a variety of general covers to be shrinkable. In particular it is shown that the shrinkability of special types of covers provide characterizations for normal and countably paracompact, normal spaces. The types of covers investigated are, weak θ-covers, weak $\bar{\theta}$-covers, point countable covers, $\delta\theta$-covers and weak $\overline{\delta\theta}$-covers. Applications of these results are answers of unsolved problems and new results for irreducible spaces.

1. Introduction.

DEFINITION 1.1. Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be an open cover of a space X. Then $\mathcal{G}$ is shrinkable if there exists a closed cover $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$.

The following is a summary of known results concerning normality and shrinkable open covers. See Alo and Shapiro [1] and Nagami [15].

THEOREM 1.2. *The following are equivalent for a space X.*

(1) *X is normal.*

(2) *Every point finite open cover of X is shrinkable.*

(3) *Every locally finite open cover of X is shrinkable.*

(4) *Every finite open cover of X is shrinkable.*

In this paper we show that normality is actually characterized by the shrinkability of several types of more general covers. These new results answer several unsolved problems as well as provide certain conditions for a space to be a "Dowker space".

In §2 the shrinkability of certain θ-type covers provides characterizations for normal and normal countably paracompact spaces. Point countable covers are studied in §3 and several results concerning irreducible spaces are obtained.

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The following results are used in the remainder of this paper.

THEOREM 1.3. *Let $\mathcal{F}$ be a closure-preserving collection of closed subsets of X such that each $F \in \mathcal{F}$ is contained in some member of the collection $\mathcal{G} = \{G_\alpha: \alpha \in A\}$. Then we may assume that $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ where $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$.*

PROOF. Assume that A is well ordered and define $F_\alpha = \bigcup\{F: F \in \mathcal{F}$ and α is the first element of A for which $F \subseteq G_\alpha\}$. It is easy to see that $\{F_\alpha: \alpha \in A\}$ has the above properties.

THEOREM 1.4. *Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be an open cover of a space X. If $\mathcal{G}$ has a refinement $\bigcup_{i=1}^\infty \mathcal{V}_i$, where $\mathcal{V}_i = \{V(\alpha, i): \alpha \in A\}$, satisfies*

(i) $\overline{V(\alpha, i)} \subseteq G_\alpha$ for each $\alpha \in A$,

(ii) $\bigcup_{\alpha \in A} V(\alpha, i)$ is a cozero set for each i, then $\mathcal{G}$ is shrinkable.

PROOF. Let $V_i^* = \bigcup_{\alpha \in A} V(\alpha, i)$ so that $\{V_i^*\}_{i=1}^\infty$ is a countable cozero cover of X. Then $\{V_i^*\}_{i=1}^\infty$ has a locally finite open refinement $\{W_i\}_{i=1}^\infty$ where $W_i \subseteq V_i^*$. Define $U(\alpha, i) = W_i \cap V(\alpha, i)$ for each $\alpha \in A$ and each i, and $H_\alpha = \bigcup_{i=1}^\infty U(\alpha, i)$. It is easy to show that $\overline{H}_\alpha \subseteq G_\alpha$ for each $\alpha \in A$ and $\{H_\alpha: \alpha \in A\}$ covers X. Hence $\mathcal{G}$ is shrinkable.

2. Shrinking of θ-covers. The authors of [7], [18], [24] have shown that θ-refinable and weakly θ-refinable spaces play an important role in the study of collectionwise normal space. Oddly enough these spaces have certain shrinkable properties for the class of normal spaces as well.

DEFINITION 2.1. A space X is *θ-refinable* if every open cover of X has a refinement $\bigcup_{i=1}^\infty \mathcal{G}_i$ satisfying

(i) Each $\mathcal{G}_i$ is an open cover of X.

(ii) Each $x \in X$ has finite positive order with respect to some $\mathcal{G}_k$. A *θ-cover* is a cover $\bigcup_{i=1}^\infty \mathcal{G}_i$ satisfying (i) and (ii) above.

REMARK. Since a θ-cover usually refines an open cover $\mathcal{U} = \{U_\alpha: \alpha \in A\}$, we can assume without loss of generality that each collection $\mathcal{G}_i$ is indexed by the set A. That is $\mathcal{G}_i = \{G(\alpha, i): \alpha \in A\}$.

DEFINITION 2.2. A space X is called *weakly θ-refinable* if every open cover of X has a refinement $\bigcup_{i=1}^\infty \mathcal{G}_i$ satisfying

(i) each $\mathcal{G}_i$ is a collection of open sets.

(ii) each $x \in X$ has finite positive order with respect to some $\mathcal{G}_k$.

DEFINITION 2.3. A space X is called *weakly $\bar{\theta}$-refinable* if every open cover of X has a refinement $\bigcup_{i=1}^\infty \mathcal{G}_i$ satisfying (i) and (ii) above and

(iii) $\{G_i^* = \bigcup[G: G \in \mathcal{G}_i]\}_{i=1}^\infty$ is point finite.

The above spaces have been studied by Bennett, Boone, Burke, Lutzer and Smith. See [3], [4], [6]–[9], [18], [19]. In [18], the author has shown that

$$\theta\text{-refinable} \Rightarrow \text{weakly } \bar{\theta}\text{-refinable} \Rightarrow \text{weakly } \theta\text{-refinable}$$

and the implications are not reversible.

We now show that weak $\bar{\theta}$-covers are shrinkable in the class of normal spaces.

CONSTRUCTION LEMMA. *Let* $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k$ *be a weak* $\bar{\theta}$*-cover of a normal space* X *with* $\mathcal{G}_k = \{G(\alpha, k): \alpha \in A\}$. *Define* $G_k^* = \bigcup_{\alpha \in A} G(\alpha, k)$ *for each* k, $\mathcal{G}^* = \{G_k^*\}_{k=1}^{\infty}$ *and*

$$P(i,j) = \left\{ \begin{array}{l} x \in X\text{: } \textit{either } \operatorname{ord}(x, \mathcal{G}^*) < i\text{; } \textit{or } \operatorname{ord}(x, \mathcal{G}^*) = i \\ \textit{and } 0 < \operatorname{ord}(x, \mathcal{G}_k) \leq j \textit{ for some } k \end{array} \right\}.$$

If $P(i,j)$ *can be covered by a collection of cozero sets* $\bigcup\{\mathcal{U}_k\}_{k=1}^{\infty} = \{U(\alpha, l): \alpha \in A, l = 1, 2, \ldots\}$ *where* $\overline{U(\alpha, l)} \subseteq G(\alpha, l)$ *for each* $\alpha \in A$ *and* $\bigcup_{\alpha \in A} U(\alpha, l)$ *is cozero, then* $P(i, j+1)$ *can be covered by such a collection.*

PROOF. Let $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k$ be a weak $\bar{\theta}$-cover of a normal space X with $\mathcal{G}_k = \{G(\alpha, k): \alpha \in A\}$. Suppose that $P(i,j)$ is covered by a collection $\mathcal{U} = \bigcup_{l=1}^{\infty} \mathcal{U}_l$ of cozero sets, where $\mathcal{U}_l = \{U(\alpha, l);\ \alpha \in A\}$, $\overline{U(\alpha, l)} \subseteq G(\alpha, l)$ for each $\alpha \in A$ and $\bigcup_{\alpha \in A} U(\alpha, l)$ is cozero.

Define: $U^* = \bigcup\{U: U \in \mathcal{U}\}$,

$$H_i = \{x \in X: \operatorname{ord}(x, \mathcal{G}^*) \leq i\} \quad (\text{closed in } X),$$

$$\mathcal{B}(j+1) = \{B: B \subseteq A, |B| = j+1\}, \text{ and for each } k,$$

$$S(k, j+1) = \{x \in X: 0 < \operatorname{ord}(x, \mathcal{G}_k) \leq j+1\}.$$

Now for each $B \in \mathcal{B}(j+1)$ define

$$F(k, i, B, j+1) = [G_k^* \cap H_i \cap S(k, j+1)] \cap [X - U^*] \cap \left[\bigcap_{\alpha \in B} G_\alpha\right]$$

and

$$\mathcal{F}_k = \{F(k, i, B, j+1): B \in \mathcal{B}(j+1)\}.$$

We assert that $\mathcal{F}_k$ is a discrete collection for each k. Let k be fixed and let $x \in X$.

(1) If $\operatorname{ord}(x, \mathcal{G}^*) < i$, then $0 < \operatorname{ord}(x, \mathcal{G}_l) < \aleph_0$ for some l; and hence $x \in P(i,j)$. Thus $x \in U^*$ and U^* is a neighborhood of x which intersects no member of $\mathcal{F}_k$.

(2) If $\operatorname{ord}(x, \mathcal{G}^*) > i$, then $X - H_i$ is a neighborhood of x which intersects no member of $\mathcal{F}_k$.

(3) Suppose $\operatorname{ord}(x, \mathcal{G}^*) = i$.

Case I: If $x \notin G_k^*$, then x belongs to exactly i members of $\mathcal{G}^*$, say $G_{\alpha_1}^*, G_{\alpha_2}^*, \ldots, G_{\alpha_i}^*$ where $k \neq \alpha_l$ for $l = 1, 2, \ldots, i$. Therefore $\bigcap_{l=1}^{i} G_{\alpha_l}^*$ is a neighborhood of x which misses $G_k^* \cap H_i$ and hence misses each member of $\mathcal{F}_k$.

Case II: Suppose $x \in G_k^*$.

(i) If $0 < \operatorname{ord}(x, \mathcal{G}_k) < j+1$, then $x \in P(i,j)$ so that U^* intersects no member of $\mathcal{F}_k$ as before.

(ii) If $\operatorname{ord}(x, \mathcal{G}_k) > j+1$, then x belongs to at least $j+2$ members of

$\mathcal{G}_k$, say $G(\alpha_1, k), G(\alpha_2, k), \ldots, G(\alpha_{j+2}, k)$. Then $\bigcap_{l=1}^{j+2} G(\alpha_l, k)$ is a neighborhood of x which misses $S(k, j+1)$ and hence intersects no member of $\mathcal{F}_k$.

(iii) If $\operatorname{ord}(x, \mathcal{G}_k) = j+1$, then x belongs to exactly $j+1$ members of $\mathcal{G}_k$, say $G(\alpha_1, k), G(\alpha_2, k), \ldots, G(\alpha_{j+1}, k)$. Hence $\bigcap_{l=1}^{j+1} G(\alpha_l, k)$ is a neighborhood of x which intersects only $F(k, i, B, j+1)$ where $B = \{\alpha_1, \alpha_2, \ldots, \alpha_j, \alpha_{j+1}\}$.

By Theorem 1.3 above, we may assume that $\mathcal{F}_k = \{F(\alpha, k): \alpha \in A\}$ for each $\alpha \in A$ and each $k \in N$. Since X is normal, there exists a collection $\mathcal{V}_k = \{V(\alpha, k): \alpha \in A\}$ of cozero sets such that

$$F(\alpha, k) \subseteq V(\alpha, K) \subseteq \overline{V(\alpha, k)} \subseteq G(\alpha, k)$$

and $\bigcup_{\alpha \in A} V(\alpha, k)$ is cozero for each k. It is easy to see that if $x \in P(i, j+1) - P(i, j)$ and is not covered by $\mathcal{U}$, then $x \in F(k, i, B, j+1)$ for some k and hence is covered by $\mathcal{V}_k$. Therefore $[\mathcal{U}] \cup [\bigcup_{k=1}^{\infty} \mathcal{V}_k]$ is the desired collection and the proof is complete.

THEOREM 2.4. *Let X be a normal space. Then every weak $\bar{\theta}$-cover of X is shrinkable.*

PROOF. Let $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k$ be a weak $\bar{\theta}$-cover of X. Let $P(i, j)$ be defined as in the Construction Lemma above for $i \geqslant 1$ and $j \geqslant 0$. Note that $P(1, 0) = \varnothing$ and $P(i, 0) = \bigcup_{l=1}^{i-1} \bigcup_{j=1}^{\infty} P(l, j)$ for $i \geqslant 2$.

By induction and the Construction Lemma above, there exists a collection $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ of cozero sets satisfying the above properties with $\mathcal{U}$ covering each $P(i, j)$. Since each $x \in X$ belongs to some $P(i, j)$, $\mathcal{U}$ covers X. Therefore by Theorem 1.4 above $\mathcal{U}$ is shrinkable and hence $\mathcal{G}$ is shrinkable.

COROLLARY 2.5. (1) *A space X is normal iff every weak $\bar{\theta}$-cover is shrinkable.*

(2) *Every open cover of a normal, weakly $\bar{\theta}$-refinable space is shrinkable.*

(3) *Every normal, weakly $\bar{\theta}$-refinable space is countably paracompact.*

PROOF. Parts (1) and (2) follow directly from Theorem 2.4. Part (3) follows from the fact that countable open covers will be shrinkable and hence X will be countably metacompact.

REMARK. Part (3) of Corollary 2.5 above shows that the condition of not being weakly $\bar{\theta}$-refinable is a necessary condition for a normal space to be a "Dowker space". That is, for a space X to be normal but not countably paracompact it must *not* be weakly $\bar{\theta}$-refinable.

Corollary 2.5 also answers in the negative Question 4 posed in **[20]** as to whether collectionwise normality is equivalent to every weak $\bar{\theta}$-cover being shrinkable.

EXAMPLE 2.6. Consider Bing's space G in **[5]**. It is easy to see that this space is weakly $\bar{\theta}$-refinable. Since it is normal, it must be countably paracompact by Corollary 2.7 and hence a non-Dowker space.

Note: Burke **[9]** has shown that this space is not θ-refinable if $|P| > 2^{\aleph_0}$ so that G is a normal, weakly $\bar{\theta}$-refinable countably paracompact space that is not θ-refinable.

EXAMPLE 2.7. In **[16]** M. E. Rudin gives an example of a collectionwise normal, noncountably paracompact space. This space of course must not be weakly $\bar{\theta}$-refinable. In fact, there exists a countable open cover of this space which is a Q-cover, that is not shrinkable.

THEOREM 2.8. *Let X be a countably paracompact normal space. Then every weak θ-cover of X is shrinkable.*

PROOF. Let $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k$ be a weak θ-cover of X where $\mathcal{G}_k = \{G(\alpha, k)\colon \alpha \in A\}$. Define $G_k^* = \bigcup_{\alpha \in A} G(\alpha, k)$ for each k. Since X is countably metacompact, $\mathcal{G}^* = \{G_k^*\}_{k=1}^{\infty}$ has a point finite open refinement $\mathcal{V}_1 = \{V(1, k)\}_{k=1}^{\infty}$ where we may assume that $V(1, 1) = G_1^*$ and $V(1, k) \subseteq G_k^*$ for $k \geqslant 2$.

Here at stage 1 we define $\mathcal{W}^1 = \bigcup_{k=1}^{\infty} \mathcal{W}_k$ where $\mathcal{W}_k = \{V(1, k) \cap G\colon G \in \mathcal{G}_k\}$. Now $\mathcal{W}^1$ is essentially a weak $\bar{\theta}$-cover of X except that $P_1 = \{x\colon x$ has finite positive order with respect to some $\mathcal{W}_k\}$ may not be all of X. Nevertheless by Theorem 2.4 above, there exists a collection $\mathcal{U}^1 = \bigcup_{k=1}^{\infty} \mathcal{U}_k^1$ of cozero sets covering P_1 such that the closure of each member of $\mathcal{U}^1$ is contained in some member of $\mathcal{G}$. We now continue by induction such that $V(1, n) = G_n^*$ at the nth-stage, constructing $\mathcal{U}^n$ as above. Then the collection $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}^n$ now covers X and has the above property. Hence $\mathcal{G}$ is shrinkable by Theorem 1.4.

COROLLARY 2.9. *A space X is normal and countably paracompact iff every weak θ-cover of X is shrinkable.*

COROLLARY 2.10. *Let X be a normal, countably paracompact weakly θ-refinable space. Then every open cover of X is shrinkable.*

Question (1). Let property S be the property that every open cover of a space X is shrinkable. For T_2 spaces we have that

paracompact $\Rightarrow$ property $S \Rightarrow$ normal + countably paracompact.

What condition is equivalent to property S?

3. Point countable covers. We now consider certain sufficient conditions for a point countable open cover to either be shrinkable or to have a minimal open refinement. Irreducible spaces, those spaces in which every open cover has a minimal open refinement, have been studied in **[6]**, **[7]**, **[10]**, **[17]**, **[19]**.

DEFINITION 3.1. (1) A collection $\{H_\alpha\colon \alpha \in A\}$ is *closure-preserving* if $\overline{\bigcup_{\beta \in B} H_\beta} = \bigcup_{\beta \in B} \overline{H_\beta}$ for every $B \subseteq A$.

(2) Let $\mathcal{U} = \{U_\alpha\colon \alpha \in A\}$ be a collection of subsets of a space X and $\leqslant$ a well-order on A. Then $\mathcal{U}$ is *linearly closure-preserving* with respect to $\leqslant$ provided every majorized subcollection of $\mathcal{U}$ is closure-preserving.

DEFINITION 3.2. A space X is said to have *property* S* if every minimal open cover of X is shrinkable to a linearly closure-preserving closed collection.

Note that every space X with property S* is normal since every finite cover of X has a minimal subcover. Furthermore property S* does not imply that X is collectionwise normal. Example G of R. H. Bing [5] is normal weak $\bar{\theta}$-refinable and countably paracompact, so that every open cover is shrinkable and has a minimal open refinement; yet it is not collectionwise normal. Also property S* is hereditary for closed sets.

We now show that property S* plays an important role in the shrinkability of point countable covers.

Let us begin with the following notion of Aull [2].

DEFINITION 3.3. A set M is *distinguished* with respect to an open cover $\mathfrak{U}$ of a space X if for distinct points $x, y \in M$ it must be true that $x \in U \in \mathfrak{U}$ implies that $y \notin U$.

REMARK. Aull [2] has shown that distinguished sets are discrete and every distinguished set is contained in a maximal (with respect to $\subseteq$) distinguished set. Also R. L. Moore has obtained a discrete set with the property of a distinguished in a different manner in *Foundations of point set theory* (Amer. Math. Soc. Colloq. Publ., vol. 13, 1962).

The following lemma is easy to prove.

LEMMA. *Let X be a T_1 space. Then every countable open cover of X has a minimal open refinement.*

We now obtain the main result for point countable covers.

THEOREM 3.4. *Let X be a T_1 space satisfying property* S*. *Then every point countable open cover of X has a minimal open refinement and hence is shrinkable.*

PROOF. Let $\mathcal{G}$ be a point countable open cover of X. Now let $M = \{x_\alpha: \alpha \in A\}$ be a maximally distinguished set in X with respect to the cover $\mathcal{G}$. Assume that A is well ordered. For each $\alpha \in A$, define $S_\alpha = \mathrm{St}(x_\alpha, \mathcal{G})$ so that $\{S_\alpha: \alpha \in A\}$ is a minimal open cover of X. Note that each S_α is a countable union of members $G(\alpha, i) \in \mathcal{G}$ since $\mathcal{G}$ is point countable. By property S*, there exists a linearly closure-preserving closed cover $\{F_\alpha: \alpha \in A\}$ of X such that $F_\alpha \subseteq S_\alpha$ for each $\alpha \in A$. By transfinite induction we construct for each $\alpha \in A$ an open (in X) collection $\mathcal{V}_\alpha = \{V(\alpha, i)\}_{i=1}^{\infty}$ satisfying:

(1) $\mathcal{V}_\alpha$ is a minimal open cover of $F_\alpha - \bigcup\{V: V \in \bigcup_{\beta<\alpha} \mathcal{V}_\beta\}$ and

(2) $\bigcup_{\beta<\alpha} \mathcal{V}_\beta$ is a minimal open cover of its union.

Step 1. Let α' be the first member of A. Since X is T_1, by the previous lemma, $\{F_{\alpha'} \cap G(\alpha', i)\}_{i=1}^{\infty}$ has an open (in $F_{\alpha'}$) refinement $\{W(\alpha', i)\}_{i=1}^{\infty}$ which covers $F_{\alpha'}$ minimally. Now let $V(\alpha', i)$ be an open set in X such that for each i, $W(\alpha', i) = V(\alpha', i) \cap F_{\alpha'}$ and $V(\alpha', i) \subseteq G(\alpha', i)$. Then $\mathcal{V}_{\alpha'} = \{V(\alpha', i)\}_{i=1}^{\infty}$ is a minimal open cover of its union which contains $F_{\alpha'}$.

Step 2. Assume that for each $\beta < \alpha$, $\mathcal{V}_\beta$ has been constructed satisfying condition (1) and (2) above.

Step 3. We now construct $\mathcal{V}_\alpha$. Define $H_\alpha = F_\alpha - \bigcup\{V: V \in \bigcup_{\beta<\alpha} \mathcal{V}_\beta\}$. As before $\{G(\alpha, i)\}_{i=1}^{\infty}$ is an open cover of H_α so that there exists an open (in X) collection $\{W(\alpha, i)\}_{i=1}^{\infty}$ which is a minimal cover of its union. Now let $V(\alpha, i) = W(\alpha, i) - \bigcup_{\beta<\alpha} F_\beta$ and $\mathcal{V}_\alpha = \{V(\alpha, i)\}_{i=1}^{\infty}$. It is easy to check that $\mathcal{V}_\beta$ satisfies conditions (1) and (2) above.

We now assert that $\mathcal{V} = \bigcup_{\alpha\in A} \mathcal{V}_\alpha$ is a minimal open refinement of $\mathcal{G}$. From the above construction $\mathcal{V}$ is a refinement of $\mathcal{G}$ and is a minimal cover of its union. Therefore we need only show that $\mathcal{V}$ covers X. Let $x \in X$ and let $\gamma \in A$ be the first index for which $x \in H_\gamma$. If x does not belong to some member of $\bigcup_{\beta<\gamma} \mathcal{V}_\beta$ then $x \in H_\gamma = F_\gamma - \bigcup\{V: V \in \bigcup_{\beta<\gamma} \mathcal{V}_\beta\}$ and hence belongs to some member of $\mathcal{V}_\gamma$. Therefore $\mathcal{V}$ covers X and the proof is complete.

COROLLARY 3.5. *Let X be a countably paracompact space satisfying property* S*. *If X is metalindelöf, then X is irreducible.*

For more general open covers we consider the $\delta\theta$-covers and weak $\delta\theta$-covers of Aull [2].

DEFINITION 3.6. A space X is called *$\delta\theta$-refinable* if every open cover of X has an open refinement $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ satisfying:

(1) each $\mathcal{G}_i$ is an open cover of X, and

(2) for each $x \in X$, $0 < \operatorname{ord}(x, \mathcal{G}_{n(x)}) \leqslant \aleph_0$ for some $n(x)$.

If each $\mathcal{G}_i$ is not required to cover X, then X is called *weakly $\delta\theta$-refinable*. Naturally, covers with the above properties are called *$\delta\theta$-covers* and *weak $\delta\theta$-covers* respectively.

In [8] J. Boone showed (Lemma 3.1) that for "sequential" spaces, the points of countable order with respect to any open cover formed a closed set. He then stated that S. Davis has observed that this result is also true when "sequential" is replaced by the property of "countable tightness".

DEFINITION 3.7. A space X is said to have *countable tightness* provided whenever $A \subseteq X$ and $x \in \bar{A}$, then there exists a countable set $C \subseteq X$ with $x \in \bar{C}$.

THEOREM 3.8 (S. DAVIS). *Let X be a space with countable tightness and $\mathcal{G}$ an open cover of X. Then the set $F = \{x: \operatorname{ord}(x, \mathcal{G}) \leqslant \aleph_0\}$ is closed.*

In these spaces we have the analogous result to Theorem 3.4 above for weak $\delta\theta$-covers.

THEOREM 3.9. *Let X be a T_1 space with countable tightness which satisfies property* S*. *Then every weak $\delta\theta$-cover of X has a minimal open refinement and hence is shrinkable.*

PROOF. Let $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ be a weak $\delta\theta$-cover of X. Then $\{F_i\}_{i=1}^{\infty}$ is a countable closed cover of X by Theorem 3.8 above where $F_i = \{x\colon \operatorname{ord}(x, \mathcal{G}_i) \leqslant \aleph_0\}$. Now $\mathcal{G}_1|F_1$ is a point countable open cover of F_1 so that by Theorem 3.4 above, there exists a collection $\mathcal{V}_1$ of open subsets of X such that $\mathcal{V}_1$ refines $\mathcal{G}_1$ and is a minimal cover of its union. Let $H_2 = F_2 - \bigcup\{V\colon V \in \mathcal{V}_1\}$ so that $\mathcal{G}_2|H_2$ is a point countable open cover of H_2. As before there exists a collection $\mathcal{V}_2$ of open subsets of X refining $\mathcal{G}_2$ such that $F_1 \cap [\bigcup\{V\colon V \in \mathcal{V}_2\}] = \varnothing$ and $\mathcal{V}_1 \cup \mathcal{V}_2$ is a minimal open cover of its union. Continuing in this fashion we construct $\mathcal{V}_i$, which refines $\mathcal{G}_i$ for each i, such that $[\bigcup_{j<i} F_j] \cap [\bigcup\{V\colon V \in \mathcal{V}_i\}] = \varnothing$ and $\bigcup_{j\leqslant i} \mathcal{V}_j$ is a minimal open cover of its union. Then it is easy to check that $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a minimal open refinement of $\mathcal{G}$.

COROLLARY 3.10. *Let X be a T_1 space with countable tightness which satisfies property S^*. If X is weakly $\delta\theta$-refinable, then X is irreducible.*

Question (1). Is Theorem 3.9 true without the condition that X have countable tightness?

Question (2). When are weak $\overline{\delta\theta}$-covers shrinkable? (Weak $\overline{\delta\theta}$-covers are weak $\delta\theta$-covers with the added condition that $\{G_i^*\}_{i=1}^{\infty}$ is point finite, where $G_i^* = \bigcup\{G\colon G \in \mathcal{G}_i\}$.)

Question (3). Is the example of M. E. Rudin [**16**] referred to in Example 2.6 above a weakly θ-refinable space? Assuming axiom †, de Caux [Topology Proc. **1** (1976), 67–78] has constructed a collectionwise normal weakly θ-refinable Dowker space.

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Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061