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ON K-PRIMITIVE RINGS

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ABSTRACT. Ortiz has defined a new radical for rings, called the K-radical, which in general lies strictly between the prime radical and the Jacobson radical. In this paper a simple internal characterization of K-primitive rings is given, and it is shown that among the K-primitive rings are prime Noetherian rings and prime rings which satisfy a polynomial identity. In addition an analogue of the density theorem is proved for K-primitive rings.

Throughout, R will denote an associative ring, not necessarily with unity element. If N is a submodule of a right R-module M, then $(N : M) = \{a \in R | Ma \subset N\}$. As in **[1]**, let K_R denote the class of all right R-modules M such that

(1) $(0 : M)$ is a prime ideal of R;

(2) if N is a submodule of M for which $(N : M) = (0 : M)$, then $N = 0$.

Ortiz has shown that the property $K_R = \varnothing$ is a radical property and that the K-radical is $\bigcap \{(0 : M) | M \in K_R\}$. A *right K-primitive* ideal of R is an ideal P such that $P = (0 : M)$ for some $M \in K_R$, and a *right K-primitive* ring is a ring in which 0 is a right K-primitive ideal. Thus R is right K-primitive if and only if R is a prime ring and K_R contains a faithful right R-module. Left K-primitive, etc., are defined analogously, and in this paper terms such as "K-primitive" will always mean "right K-primitive".

PROPOSITION 1. *A prime ring R is K-primitive if and only if R contains a right ideal I which is maximal with respect to the property* $(I : R) = 0$.

PROOF. Suppose R is K-primitive and let M be a faithful right R-module in K_R. Choose $m \neq 0$ in M; let $I = \{x \in R | mx = 0\}$; and suppose $a \in (I : R)$. If $N = \{n \in M | nRa = 0\}$, then since $m \in N$, N is a nonzero submodule of M, whence there exists $b \neq 0$ in $(N : M)$. Thus $Mb \subset N$ and so $MbRa = 0$, which yields $bRa = 0$ and hence $a = 0$ since R is prime. Therefore R is a right ideal of R satisfying $(I : R) = 0$. Now let J be any right ideal of R properly containing I. Since mJ is a nonzero submodule of M, there is an $x \neq 0$ in $(mJ : M)$. Let $r \in R$. Since $mRx \subset Mx \subset mJ$, there exists $y \in J$

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such that $mrx = my$. Thus $rx - y \in I$ and so $rx = (rx - y) + y \in J$. Hence $Rx \subset J$, and so I is maximal with respect to $(I : R) = 0$.

Conversely, assume I is a right ideal of R which is maximal with respect to $(I : R) = 0$, and let M be the right R-module R/I. Since $(0 : M) = (I : R) = 0$, M is faithful. If N is a nonzero submodule of M, then its inverse image J in R is a right ideal properly containing I, and by the maximality of I, $(J : R) \neq 0$. Since $(N : M) = (J : R)$, we have $M \in K_R$, and so R is K-primitive.

COROLLARY 1. *Every primitive ring is K-primitive.*

COROLLARY 2. *Every right Noetherian prime ring is K-primitive.*

THEOREM 1. *If R is a right order in a simple Artinian ring Q with center F such that $Q = RF$, then R is K-primitive.*

PROOF. $Q \cong D_n$ for some division ring D. Let V be an n-dimensional right vector space over D; choose $v \neq 0$ in V; and let $I = \{x \in R | vx = 0\}$. If $a \in (I : R)$, then

$$Va = vQa = vRFa = vRaF \subset vIF = 0,$$

whence $a = 0$. Thus I is a right ideal of R satisfying $(I : R) = 0$. Let J be a right ideal of R properly containing I. Since vJ is a nonzero R-submodule of V, vJF is a nonzero Q-submodule of V and hence $V = vJF$. Choose $x \in JF$ such that $v = vx$. There exist a and b in R with b regular such that $1 - x = ab^{-1}$. Hence $vab^{-1} = v(1 - x) = 0$, and so $va = 0$, which means that $a \in I$. Thus $b = a + xb \in I + JF \subset JF$. Since JF is a right ideal of Q and contains an invertible element b of Q, we have $JF = Q$. We now write $1 = \Sigma c_i\lambda_i$ where the c_i are in J and the λ_i are in F. We may also write $\lambda_i = d_ie^{-1}$ where the d_i are in R and e is a regular element of R. For any $r \in R$ we have

$$r = 1 \cdot r = \sum c_i\lambda_i r = \sum c_i r\lambda_i = \sum c_i r d_i e^{-1},$$

whence $re = \Sigma c_i r d_i \in J$. Thus $Re \subset J$ and so $(J : R) \neq 0$. Therefore I is maximal with respect to $(I : R) = 0$, so by Proposition 1, R is K-primitive. □

Using Theorem 1 and Posner's Theorem [2], we establish

COROLLARY 3. *Every prime ring satisfying a polynomial identity over its centroid is both left and right K-primitive.*

A special case of Theorem 1 is worth noting. If Q is a division ring–that is, if R is a right Ore domain–then the proof of Theorem 1 shows that the right ideal I is 0. We shall call a K-primitive ring *strongly K-primitive* in case the right ideal I which is maximal with respect to $(I : R) = 0$ is $I = 0$. Thus we have

COROLLARY 4. *If R is a right Ore domain with right quotient ring D having center F such that $D = RF$, then R is strongly K-primitive.*

Several obvious questions arise concerning K-primitive rings:

1. Are left and right K-primitivity equivalent?

2. Converse of Corollary 4. Is every strongly K-primitive ring a right Ore domain which, together with the center of its right quotient ring D, generates D?

3. Is every right Ore domain K-primitive? Strongly K-primitive?

4. Can the hypothesis $Q = RF$ in Theorem 1 be removed? Equivalently, is every prime right Goldie ring K-primitive?

These questions are open except for the second part of 3: Any simple right Noetherian domain, not a division ring, is a right Ore domain but is not strongly K-primitive; it is K-primitive however, by Corollary 2. Also, question 2 can be answered partially by

PROPOSITION 2. *Every strongly K-primitive ring is a right Ore domain.*

PROOF. Let a and b be nonzero elements of the strongly K-primitive ring R. By maximality of the zero right ideal, every nonzero right ideal I of R satisfies $(I : R) \neq 0$. In particular $(bR : R) \neq 0$, so there exists $b_1 \neq 0$ in R such that $Rb_1 \subset bR$. If $ab = 0$, then $aRb_1 \subset abR = 0$, a contradiction since R is prime; thus R has no zero divisors. Moreover, since $ab_1 \in Rb_1 \subset bR$, there exists $a_1 \neq 0$ in R such that $ab_1 = ba_1$, so R is a right Ore domain. $\square$

In [1] Ortiz showed that every K-primitive ring can be embedded in a full ring of linear transformations of a vector space over a division ring. We shall investigate this embedding and in fact prove an analogue of Jacobson's density theorem for K-primitive rings. One formulation of density is the following: If V is a vector space over a division ring D, then a subring R of $\mathrm{Hom}_D(V, V)$ is dense if and only if V is irreducible and for every finite-dimensional subspace W and every vector $u \notin W$, $(u(0 : W) : V) = R$. A slight variation of this definition leads to the desired characterization of K-primitive rings. We first define V to be *K-irreducible* if and only if $vRa = 0$ with $v \in V$ and $a \in R$ implies $v = 0$ or $a = 0$. A subring R of $\mathrm{Hom}_D(V, V)$ will be called *K-dense* if and only if V is K-irreducible and for every finite-dimensional subspace W and every vector $u \notin W$, $(u(0 : W) : V) \neq 0$.

THEOREM 2. *If R is a K-primitive ring, V is a faithful module in K_R, and $\bar{V}$ is the quasi-injective hull of V, then $D =$ $\mathrm{Hom}_R(\bar{V}, \bar{V})$ is a division ring, V is a vector space over D, and R is a K-dense subring of* $\mathrm{Hom}_D(V, V)$. *Conversely, if a ring R is a K-dense subring of* $\mathrm{Hom}_D(V, V)$ *for some vector space V over a division ring D, then R is K-primitive, $V \in K_R$, and $D =$* $\mathrm{Hom}_R(\bar{V}, \bar{V})$.

PROOF. Assume first that R is a K-primitive ring with V a faithful module in K_R. Ortiz [1] has shown that D is a division ring, that V is a vector space over D, and that the mapping $a \to a'$ defined by $va' = va$ for $v \in V$ and $a \in R$ is an embedding of R in $\mathrm{Hom}_D(V, V)$. We must show that R is K-dense. Let $a \neq 0$ be in R and let $N = \{v \in V | vRa = 0\}$. N is a submodule of V and if $N \neq 0$, then $(N : V) \neq 0$, whence there exists $b \neq 0$ in R such that $VbRa \subset NRa = 0$. This implies that $bRa = 0$, a contradiction since R is prime. Thus $N = 0$, that is, V is K-irreducible. If we show that for every finite-dimensional subspace W and every vector $u \notin W$, we have $u(0 : W) \neq 0$, then $u(0 : W)$, being a nonzero submodule of V, would satisfy $(u(0 : W) : V) \neq 0$, thereby proving that R is K-dense. Suppose then that W is a finite-dimensional subspace of smallest dimension for which there is a vector $u \notin W$ such that $u(0 : W) = 0$. If $W = 0$, then $u(0 : W) = uR \neq 0$, so $\dim W > 0$. Let $W = W_0 + wD$ where

$$\dim W_0 = \dim W - 1$$

and $w \notin W_0$. The mapping T: $w(0 : W_0) \to u(0 : W_0)$ defined by $(wa)T = ua$ for $a \in (0 : W)$ is well-defined since $wa_1 = wa_2$ with a_1, a_2 in $(0 : W_0)$ implies

$$a_1 - a_2 \in (0 : W_0) \cap (0 : w) = (0 : W)$$

and hence $u(a_1 - a_2) = 0$. Since $\bar{V}$ is quasi-injective, T can be extended to $\lambda \in \operatorname{Hom}_R(\bar{V}, \bar{V}) = D$. For any $a \in (0 : W_0)$ we have

$$ua = (wa)T = (wa)\lambda = (w\lambda)a$$

and so $(u - w\lambda)(0 : W_0) = 0$. By minimality of W we must have $u - w\lambda \in W_0$ and hence $u \in W_0 + wD = W$, a contradiction. This proves that R is K-dense.

Conversely, assume that R is a K-dense subring of $\operatorname{Hom}_D(V, V)$ for some vector space V over a division ring D. Suppose A and B are left ideals of R such that $AB = 0$. Choose $a \neq 0$ in A and $v \neq 0$ in V. For any $b \in B$ we have $vRaRb \subset vAB = 0$. The K-irreducibility of V implies that $vRa \neq 0$, so choosing $r \in R$ such that $vra \neq 0$, we have $vraRb = 0$ and by K-irreducibility again we have $b = 0$. Thus R is a prime ring. Suppose N is a nonzero R-submodule of V and choose $u \neq 0$ in N. Taking $W = 0$ we have

$$0 \neq (u(0 : W) : V) = (uR : V) \subset (N : V)$$

and hence $V \in K_R$ and R is K-primitive. To show that $D = \operatorname{Hom}_R(\bar{V}, \bar{V})$, let $\lambda \in D$. The mapping $v \to v\lambda$ can be extended to $\lambda' \in \operatorname{Hom}_R(\bar{V}, \bar{V})$. We shall show that λ' is unique and that identifying λ with λ' yields the desired conclusion. Suppose λ' and λ'' are both extensions of $v \to v\lambda$. Then $\lambda' - \lambda''$ is in $\operatorname{Hom}_R(\bar{V}, \bar{V})$ and since the latter is a division ring, either $\lambda' - \lambda''$ is one-to-one or $\lambda' - \lambda''$ is 0. Since

$$\ker(\lambda' - \lambda'') \supset V \neq 0,$$

we have $\lambda' = \lambda''$. Thus the identification of λ with λ' is well defined and embeds D in $\operatorname{Hom}_R(\bar{V}, \bar{V})$. To complete the proof we must show that every element of $\operatorname{Hom}_R(\bar{V}, \bar{V})$ is λ' for some $\lambda \in D$. Suppose $f \neq 0$ is in $\operatorname{Hom}_R(\bar{V}, \bar{V})$; since f is one-to-one, $0 \neq Vf \subset V$. Thus there exist u, w in V such that $u = wf \neq 0$. Suppose u and w are linearly independent over D.

Then $(u(0:w):V) \neq 0$ so there exists $a \neq 0$ in R such that

$$Va \subset u(0:w) = (wf)(0:w) = w(0:w)f = 0,$$

a contradiction. Hence u and w are linearly dependent, say $u = w\lambda$ for some $\lambda \in D$. Then $w(\lambda' - f) = 0$ and hence $\lambda' - f$, not being one-to-one, must be 0.

References

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