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EXTREME POINTS OF SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. We determine coefficient bounds, distortion and covering theorems, and the extreme points for various subclasses of close-to-convex functions. All results are sharp.

1. Introduction. Let $\mathcal{S}$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $\mathcal{U}$. A normalized function f is said to be close-to-convex if there exists a function

$$g(z) = b_1 z + \cdots \qquad (\operatorname{Re} b_1 > 0) \tag{1}$$

starlike with respect to the origin for which

$$\operatorname{Re}\{zf'/g\} > 0 \qquad (z \in \mathcal{U}). \tag{2}$$

It is well known [3] that the close-to-convex functions, denoted by $\mathcal{C}$, are contained in $\mathcal{S}$.

In this paper we investigate distortion properties, coefficient bounds, and the extreme points of several subclasses of $\mathcal{C}$. A function f is said to be in $\mathcal{C}_1$ if there exists a convex function g of the form (1) such that (2) is satisfied. If there exists such a g satisfying

$$\operatorname{Re}\{[zf']'/g'\} > 0 \qquad (z \in \mathcal{U})$$

then f is said to be in $\mathcal{C}_2$. If

$$\operatorname{Re}\{[z[zf']']'/[zg']'\} > 0 \qquad (z \in \mathcal{U}),$$

then f is said to be in $\mathcal{C}_3$. In relating these classes to one another, we will rely on the following lemma due to Sakaguchi [4].

LEMMA A. *Let* $F(z) = z + \cdots$ *be analytic and* $G(z) = b_1 z + \cdots$ *be analytic and starlike in* $\mathcal{U}$ *with* $\operatorname{Re} b_1 > 0$. *If* $\operatorname{Re} F'/G' > 0$ $(z \in \mathcal{U})$, *then* $\operatorname{Re} F/G > 0$ $(z \in \mathcal{U})$.

Since g convex implies zg' is starlike, an application of Lemma A shows that $\mathcal{C}_3 \subset \mathcal{C}_2$. Reapplying Lemma A we see that $\mathcal{C}_2 \subset \mathcal{C}_1$. Further $\mathcal{C}_1 \subset \mathcal{C}$

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because convex functions are starlike. We thus have the inclusion relations $\mathcal{C}_3 \subset \mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{C}$.

Geometrically, a function f is in the family $\mathcal{C}_2$ if zf' maps each circle $z = re^{i\theta}$ $(r < 1)$ onto a simple closed curve whose unit tangent vector never drops back on itself more than π radians as θ increases. That is, $f \in \mathcal{C}_2$ if and only if $zf' \in \mathcal{C}$. The family $\mathcal{C}_1$, while a proper subclass of the close-to-convex functions, is not contained in the family of starlike functions. In fact, there exist functions in $\mathcal{C}_2$ that are not starlike. For example, the function

$$h(z) = \frac{1-i}{2}\,\frac{z}{1-z} - \frac{1+i}{2}\log(1-z)$$

is shown in the next section to be in $\mathcal{C}_2$. However for ε sufficiently small, $\mathrm{Re}(zh'(z)/h(z)) < 0$ when $z = e^{i\theta}$, $-\varepsilon < \theta < 0$.

2. Extreme points of $\mathcal{C}_1$ and $\mathcal{C}_2$. For a compact family $\mathcal{F}$, we denote the closed convex hull of $\mathcal{F}$ by cl co $\mathcal{F}$ and the extreme points of cl co $\mathcal{F}$ by $\mathcal{E}$(cl co $\mathcal{F}$).

THEOREM 1. *Let X be the torus $\{(x,y)|x| = |y| = 1\}$, P be the set of probability measures on X,*

$$k(z,x,y) = (1+x)\frac{z}{1-yz} + x\bar{y}\log(1-yz),$$

where $z \in \mathcal{U}$ and $|x| = |y| = 1$, and let $\mathcal{F}$ be the set of functions f_μ defined by

$$f_\mu(z) = \int_X k(z,x,y)\,d\mu(x,y), \qquad \mu \in \mathcal{P}.$$

Then

$$\text{cl co }\mathcal{C}_1 = \mathcal{F}$$

and

$$\mathcal{E}(\text{cl co }\mathcal{C}_1) = \{k(z,x,y)|x \neq -1\}.$$

PROOF. Our proof will follow along the lines of the proof for $\mathcal{E}$(cl co $\mathcal{C}$), found in **[1]**. We first show that cl co $\mathcal{C}_1 \subset \mathcal{F}$. If $f \in \mathcal{C}_1$, then $p(z) = zf'(z)/g(z)$ has positive real part in $\mathcal{U}$ for some convex function g. By Herglotz' theorem, we can express $p(z)$ as

$$p(z) = \int_\Gamma \frac{p(0)u + \overline{p(0)}\,z}{u-z}\,d\alpha(u) \tag{3}$$

for some α a probability measure on the unit circle Γ. In **[1]** it is shown that we can express $g(z)$ as

$$g(z) = \int_\Gamma \frac{g'(0)z}{1-vz}\,d\beta(v), \tag{4}$$

where β is also a probability measure on Γ. Since $g'(0)p(0) = 1$, we use (3), (4) and Fubini's theorem to obtain

$$f'(z) = \int_\Gamma \frac{u + g'(0)\overline{p(0)}\,z}{u - z}\, d\alpha(u) \cdot \int_\Gamma \frac{1}{1 - vz}\, d\beta(v)$$

$$= \int_X \frac{1 + \varepsilon \bar{u} z}{(1 - \bar{u}z)(1 - vz)}\, d\alpha(u)\, d\beta(v), \tag{5}$$

where $\varepsilon = \overline{p(0)}g'(0)$ satisfies $|\varepsilon| = 1$. To show that $f \in \mathcal{F}$ it is sufficient to show that the kernel functions in (5) belong to $\mathcal{F}'$, the set of derivatives of functions belonging to $\mathcal{F}$. By a theorem in **[1]**, given u and v there is a probability measure γ on Γ such that

$$\frac{1 + \varepsilon \bar{u} z}{(1 - \bar{u}z)(1 - vz)} = \int_\Gamma \frac{1 + \varepsilon \bar{u} z}{(1 - wz)^2}\, d\gamma(w).$$

Thus we need only show for arbitrary w, $|w| = 1$, that we can find x, y, $|x| = |y| = 1$, such that

$$\frac{d}{dz}\, k(z, x, y) = \frac{1 + xyz}{(1 - yz)^2} = \frac{1 + \varepsilon \bar{u} z}{(1 - wz)^2}.$$

Choosing the unit point mass $k(z, x, y) = k(z, \varepsilon \bar{w} \bar{u}, w)$, we see that cl co $\mathcal{C}_1 \subset \mathcal{F}$.

To show that $\mathcal{F} \subset$ cl co $\mathcal{C}_1$, we need only show that $\{k(z, x, y)\} \subset \mathcal{C}_1$ for $|x| = |y| = 1$. Choose a complex number $\delta = \delta(x)$ so that $\operatorname{Re}\{\delta(1 + xyz)/(1 - yz)\} > 0$. Since $g(z) = z/\delta(1 - yz)$ is convex, we have

$$\operatorname{Re} \frac{z\,dk(z, x, y)/dz}{g(z)} = \operatorname{Re} \frac{\delta(1 + xyz)}{1 - yz} > 0,$$

which shows that $\{k(z, x, y)\} \subset \mathcal{C}_1$.

Thus the only possible extreme points for $\mathcal{C}_1$ are $\{k(z, x, y)\}$. Taking $g = f$ in the definition of $\mathcal{C}_1$ and noting that convex functions are starlike, we see that $\mathcal{C}_1$ contains the convex functions. Since $k(z, -1, y) = -\bar{y}\log(1 - yz)$ is convex but is not an extreme point of the closed convex hull of convex functions, it cannot be an extreme point of the larger set cl co $\mathcal{C}_1$.

Excluding $x_0 = -1$ from consideration, it suffices to show that for each x_0, y_0, $|x_0| = |y_0| = 1$,

$$k(z, x_0, y_0) = \int_X k(z, x, y)\, d\mu(x, y) \tag{6}$$

is possible only if μ is a unit point mass at (x_0, y_0). Differentiating both sides of (6) with respect to z, we obtain

$$\frac{1 + x_0 y_0 z}{(1 - y_0 z)^2} = \int_X \frac{1 + xyz}{(1 - yz)^2}\, d\mu(x, y).$$

Setting $z = \bar{y}_0 r$ and letting $r \to 1^-$, we have

$$1 + x_0 = \lim_{r \to 1^-} \int_X \left(\frac{1 - r}{1 - y\bar{y}_0 r} \right)^2 (1 + xy\bar{y}_0 r)\, d\mu(x, y). \tag{7}$$

Since the integrand in (7) is bounded by 2, we may apply the Lebesgue bounded convergence theorem to obtain

$$1 + x_0 = \int_{\Gamma \times \{y_0\}} (1 + x)\, d\mu(x, y).$$

Setting $\Gamma_0 = \Gamma \times \{y_0\}$ and $a = \mu(\Gamma_0)$, we have $0 \leqslant a \leqslant 1$ and

$$1 + x_0 = a + \int_{\Gamma_0} x\, d\mu(x, y). \tag{8}$$

Since $|x_0 + (1 - a)| = |\int_{\Gamma_0} x\, d\mu(x, y)| \leqslant a$ and $|x_0 + (1 - a)| \geqslant |x_0| - (1 - a) = a$, we must have $x_0 = -1$ or $a = 1$. Since $x_0 \neq -1$, it follows that $a = 1$. From (8) we have $x_0 = \int_{\Gamma_0} x\, d\mu(x, y)$, which can hold only if μ is a unit point mass at (x_0, y_0).

COROLLARY 1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_1$, *then* $|a_n| \leqslant 2 - 1/n$, *with equality for* $k(z, 1, -1)$.

PROOF. We need only consider $f \in \mathcal{C}_1$ of the form $k(z, x, y)$. It is easy to see that the modulus of the coefficients of k are maximized when $x = 1$ and $y = -1$.

Similarly we have

COROLLARY 2. *If* $f \in \mathcal{C}_1$, *then*

$$\frac{2r}{1 + r} - \log(1 + r) \leqslant |f(z)| \leqslant \frac{2r}{1 - r} + \log(1 - r) \qquad (|z| \leqslant r)$$

and

$$\frac{1 - r}{(1 + r)^2} \leqslant |f'(z)| \leqslant \frac{1 + r}{(1 - r)^2} \qquad (|z| \leqslant r),$$

with equality for $k(z, 1, 1)$ *at* $z = \pm r$.

We can use similar arguments to determine the extreme points of $\mathcal{C}_2$. But we will use known results for $\mathcal{C}$ to give a quicker proof.

THEOREM 2. *Let* X *be the torus* $\{(x, y) \mid |x| = |y| = 1\}$, $\mathcal{P}$ *be the set of probability measures on* X,

$$h(z, x, y) = \frac{1 - x\bar{y}}{2} \frac{z}{1 - yz} - \frac{1 + x\bar{y}}{2} \bar{y} \log(1 - yz)$$

for $|x| = |y| = 1$, *and let* $\mathcal{F}$ *be the set of functions* f_μ *defined by*

$$f_\mu(z) = \int_X h(z, x, y)\, d\mu(x, y) \qquad (\mu \in \mathcal{P}).$$

Then

$$\operatorname{cl}\operatorname{co}\mathcal{C}_2 = \mathcal{F}$$

and

$$\mathcal{E}(\operatorname{cl}\operatorname{co}\mathcal{C}_2) = \{h(z, x, y) | x \neq y\}.$$

PROOF. Observe that $f \in \mathcal{C}_2$ if and only if $zf' \in \mathcal{C}$. Thus the operator L defined by $L(f) = \int_0^z f(\zeta)/\zeta \, d\zeta$ is a linear homeomorphism on the space of analytic functions with $L(\mathcal{C}) = \mathcal{C}_2$. Since

$$h(z, x, y) = \int_0^z \frac{1 - (x + y)\zeta/2}{(1 - y\zeta)^2} \, d\zeta,$$

the result follows from the results for $\mathcal{C}$ proved in [**1**].

COROLLARY. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_2$, *then* $|a_n| \leqslant 1$ *and*

$$\frac{r}{1 + r} \leqslant |f(z)| \leqslant \frac{r}{1 - r} \qquad (|z| \leqslant r),$$

$$\frac{1}{(1 + r)^2} \leqslant |f'(z)| \leqslant \frac{1}{(1 - r)^2} \qquad (|z| \leqslant r).$$

Equality in all cases is obtained for $f(z) = z/(1 - z)$.

REMARKS. The extreme points of both $\mathcal{C}_1$ and $\mathcal{C}_2$ are linear combinations of the extreme points of the convex functions and the functions convex of order $\frac{1}{2}$. See [**2**]. Setting $x = -y$, we see that the extreme points of convex functions are contained in those for $\mathcal{C}_2$.

3. The class $\mathcal{C}_3$. The standard techniques cannot be applied to determine the extreme points of $\operatorname{cl}\operatorname{co}\mathcal{C}_3$ because of the presence of an additional parameter in the numerator. Nevertheless we still have sharp coefficient bounds and distortion theorems for the class $\mathcal{C}_3$.

THEOREM 3. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_3$, *then* $|a_n| \leqslant 2/3 + 1/3n^2$. *This result is sharp, with equality for*

$$f(z) = \frac{2}{3}\frac{z}{1 - z} - \frac{1}{3}\int_0^z \frac{\log(1 - \zeta)}{\zeta} \, d\zeta.$$

PROOF. If $f \in \mathcal{C}_3$, then there exists a convex function $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and a function of positive real part $p(z) = \sum_{n=0}^{\infty} c_n z^n$ with $|b_1| = |c_0| = 1$ such that $[z[zf']']' = [zg']'p$. Then

$$[z[zf']']' = \sum_{n=1}^{\infty} n^3 a_n z^{n-1} = \left(\sum_{n=1}^{\infty} n^2 b_n z^{n-1}\right)\left(\sum_{n=0}^{\infty} c_n z^n\right).$$

Equating coefficients, we have $n^3 a_n = \sum_{k=1}^{n} k^2 b_k c_{n-k}$. It is well known that $|b_n| \leqslant 1$ and $|c_n| \leqslant 2$ for $n \geqslant 1$. Hence

$$n^3|a_n| \leqslant 2\sum_{k=1}^{n-1} k^2 + n^2 = \frac{n(n - 1)(2n - 1)}{3} + n^2,$$

which simplifies to $|a_n| \leqslant \frac{2}{3} + 1/3n^2$. To show that the extremal function is in $\mathcal{C}_3$, we take $g(z) = z/(1 - z)$.

THEOREM 4. *If $f \in \mathcal{C}_3$, then*

$$\frac{2}{3}\,\frac{r}{1+r} + \frac{1}{3}\int_0^r \frac{\log(1+t)}{t}\,dt \leqslant |f(z)| \leqslant \frac{2}{3}\,\frac{r}{1-r} - \frac{1}{3}\int_0^r \frac{\log(1-t)}{t}\,dt \qquad (|z| \leqslant r),$$

$$\frac{2}{3}\,\frac{1}{(1+r)^2} + \frac{1}{3}\,\frac{\log(1+r)}{r} \leqslant |f'(z)| \leqslant \frac{2}{3}\,\frac{1}{(1-r)^2} - \frac{1}{3}\,\frac{\log(1-r)}{r} \qquad (0 < |z| \leqslant r).$$

Equality holds in all cases for the extremal function of Theorem 3.

PROOF. Setting $h = zf'$, we may write $[zh']' = pg'$, where $p(z)$ is a function of positive real part, $g(z)$ is a starlike function, and $|p(0)| = |g'(0)| = 1$. It is well known that $(1 - r)/(1 + r) \leqslant |p(z)| \leqslant (1 + r)/(1 - r)$ and $(1 - r)/(1 + r)^3 \leqslant |g'(z)| \leqslant (1 + r)/(1 - r)^3$ for $|z| \leqslant r$. Hence

$$\frac{(1-r)^2}{(1+r)^4} \leqslant |[zh'(z)]'| \leqslant \frac{(1+r)^2}{(1-r)^4} \qquad (|z| \leqslant r). \tag{9}$$

Integrating along the straight line segment from the origin to $z = re^{i\theta}$ in the right inequality of (9) we obtain

$$|zh'(z)| \leqslant \int_0^r \frac{(1+t)^2}{(1-t)^4}\,dt = \frac{3r + r^3}{3(1-r)^3} \qquad (|z| = r). \tag{10}$$

Now for every r choose z_0, $|z_0| = r$, such that $|h'(z_0)| = \min_{|z|=r}|h'(z)|$. If $L(z_0)$ is the pre-image of the segment $\{0, z_0h'(z_0)\}$, then

$$\begin{aligned} |zh'(z)| \geqslant |z_0h'(z_0)| &= \int_{L(z_0)} |(zh'(z))'|\,|dz| \\ &\geqslant \int_0^r \frac{(1-t)^2}{(1+t)^4}\,dt = \frac{3r + r^3}{3(1+r)^3}. \end{aligned} \tag{11}$$

In view of (10) and (11),

$$\frac{3 + r^2}{3(1+r)^3} \leqslant |[zf'(z)]'| \leqslant \frac{3 + r^2}{3(1-r)^3} \quad (|z| = r).$$

Using again the method that gave us (10) and (11), we obtain

$$\frac{2}{3}\,\frac{r}{(1+r)^2} + \frac{1}{3}\log(1+r) \leqslant |zf'(z)| \leqslant \frac{2}{3}\,\frac{r}{(1-r)^2} - \frac{1}{3}\log(1-r).$$

One more application yields

$$\frac{2}{3} \frac{r}{1+r} + \frac{1}{3} \int_0^r \frac{\log(1+t)}{t}\, dt$$
$$\leqslant |f(z)| \leqslant \frac{2}{3} \frac{r}{1-r} - \frac{1}{3} \int_0^r \frac{\log(1-t)}{t}\, dt.$$

The coefficient bounds give some indication as to the degree of containment of $\mathcal{C}_3 \subset \mathcal{C}_2 \subset \mathcal{C}_1$. Another measure is the following covering theorem.

THEOREM 5. *The disk* $\mathcal{U}$ *is mapped onto a domain that contains the disk* $|w| < 1 - \log 2 \approx 0.31$ *by any* $f \in \mathcal{C}_1$, *onto a domain that contains the disk* $|w| < 0.50$ *by any* $f \in \mathcal{C}_2$, *and onto a domain that contains the disk* $|w| < (\pi^2 + 12)/36 \approx 0.61$ *by any* $f \in \mathcal{C}_3$.

PROOF. Let $r \to 1^-$ in the lower bound of the distortion results for f in the three classes.

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