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AN INTEGRODIFFERENTIAL EQUATION ASYMPTOTICALLY OF CONVOLUTION TYPE

KENNETH B. HANNSGEN

ABSTRACT. The resolvent formula is used to study the asymptotic behavior ($t \to \infty$) of solution to integrodifferential equations which are close in some sense to equations of convolution type with integrable resolvents.

I. Introduction. For the problem

$$x'(t) + \int_0^t b(t, s)x(s)\, ds = f(t) \tag{1.1}$$

($' = d/dt$, $t \in \mathbf{R}^+ \equiv [0, \infty)$) with initial condition $x(0) = x_0$, we give conditions on b which ensure that $x \in L^p(\mathbf{R}^+)$ if $f \in L^p(\mathbf{R}^+)$, for some $p \geqslant 1$. We shall assume that, for large t and s, $b(t, s)$ is close to a kernel $a(t - s)$ of convolution type with resolvent r in $L^1(\mathbf{R}^+)$. We shall also present some results for related almost linear problems.

Throughout this paper, $\|\varphi\|$ and $\|\varphi\|_p$ denote respectively the L^1 and L^p norms of the function φ: $\mathbf{R}^+ \to \mathbf{R}$. A *solution* of (1.1) is a locally absolutely continuous function x: $\mathbf{R}^+ \to \mathbf{R}$ such that (1.1) holds almost everywhere.

If $b(t, s) = a(t - s)$ $(0 < s < t)$ with a locally L^1 on $\mathbf{R}^+$ ("$a \in LL^1(\mathbf{R}^+)$"), then, for $f \in LL^1(\mathbf{R}^+)$,

$$x(t) = x_0 r(t) + \int_0^t r(t - s)f(s)\, ds \qquad (0 \leqslant t < \infty), \tag{1.2}$$

where r, the (differential) resolvent of a, is the solution of

$$r'(t) + \int_0^t a(t - s)r(s)\, ds = 0, \qquad r(0) = 1. \tag{1.3}$$

(See [1], for example.) Thus $x \in L^p(\mathbf{R}^+)$ for all $f \in L^p(\mathbf{R}^+)$ $(1 \leqslant p \leqslant \infty)$ if

$$r \in L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+). \tag{1.4}$$

Assuming (1.4), we employ (1.2) and some simple estimates to derive our results for (1.1) with the more general kernel $b(t, s)$. Among previous studies of stability theory for integrodifferential equations, involving the resolvent formula, we mention those of S. I. Grossman and R. K. Miller [1], [2] and of Miller [7].

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II. Linear equations. Our first result displays the method in its simplest form.

THEOREM 2.1. *Let* $1 \leqslant p \leqslant \infty$. *Let* $b \in LL^1(S)$, *where* $S = \{0 \leqslant s \leqslant t < \infty\}$, *and suppose there exists* $a \in LL^1(\mathbf{R}^+)$, *with resolvent* r *satisfying* (1.4), *such that for each* $\varepsilon > 0$ *there exist* $T > 0$ *and* $c \in L^1(\mathbf{R}^+)$ *with* $\|c\| < \varepsilon$ *and*

$$|b(t + T, s + T) - a(t - s)| \leqslant c(t - s) \quad a.e. \text{ in } S, \tag{1.5}$$

$$\int_0^\infty \left[\int_0^T |b(t + T, s)|\, ds \right]^p dt < \infty. \tag{1.6}$$

Let $f \in L^p(\mathbf{R}^+)$, *and let* x *be a solution of* (1.1). *Then* $x \in L^p(\mathbf{R}^+)$.

We discuss and illustrate our results in §V; for example, we show that the hypotheses of Theorem 2.1 hold with $p = 1$ for a large class of $a \in L^1(\mathbf{R}^+)$ with $b(t, s) = \alpha(t)\beta(s)a(t - s)$, where $\alpha(t) \to 1$, $\beta(t) \to 1$ as $t \to \infty$. On the other hand, with $f \equiv 0$, $b(t, s) = a(t - s) + \beta(s)A(t - s)$ ($\beta(s) = 0$ for $s > 1$) it can happen that $x(\infty) = 0$ but $x \notin L^1(\mathbf{R}^+)$. The following positive result holds, however.

THEOREM 2.2. *Let* x *be a solution of* (1.1) *with*

$$b(t, s) = \alpha(t)\beta(s)A(t - s) + \gamma(s)a(t - s),$$

where

$$a \in LL^1(\mathbf{R}^+) \quad \textit{with resolvent } r \in L^1(\mathbf{R}^+),\ r' \in L^1(\mathbf{R}^+), \tag{1.7}$$

$$0 < A \in LL^1(\mathbf{R}^+), \quad \alpha(t) \downarrow 0 \quad (t \uparrow \infty), \quad \alpha A \in L^1(\mathbf{R}^+), \tag{1.8}$$

$$\beta, \gamma \in L^\infty(\mathbf{R}^+), \quad \gamma(t) \to 1 \quad (t \to \infty), \tag{1.9}$$

$$f \in L^1(\mathbf{R}^+). \tag{1.10}$$

Then $x \in L^1(\mathbf{R}^+)$.

Well-known sufficient conditions for (1.7) are discussed in §V.

III. Almost linear equations. The results of §II, together with a perturbation theorem of S. I. Grossman and R. K. Miller [**1**, Theorem 4], immediately yield an existence-stability result for the almost linear equation

$$x'(t) + \int_0^t b(t, s)x(s)\, ds + (hx)(t) = f(t), \qquad x(0) = x_0, \tag{3.1}$$

where h: $L^p \to L^p$ is of higher order with respect to L^p. (*Higher order* means $h0 = 0$ and $\|h\varphi_1 - h\varphi_2\|_p = o\|\varphi_1 - \varphi_2\|_p$ as $\|\varphi_1\|_p$, $\|\varphi_2\|_p \to 0$. *Solution* is defined as for (1.1).)

COROLLARY 3.1. *Let* b *satisfy the conditions of Theorem* 2.1 [*Theorem* 2.2], *and let* h *be of higher order with respect to* $L^p(\mathbf{R}^+)$ [$L^1(\mathbf{R}^+)$]. *Then for each* $\varepsilon > 0$, *there exists a number* $\eta > 0$ *such that if* $|x_0| \leqslant \eta$ *and* $\|f\|_p \leqslant \eta$ [$\|f\| \leqslant \eta$], *then* (3.1) *has a unique solution in* $L^p(\mathbf{R}^+)$ [$L^1(\mathbf{R}^+)$] *with* $\|x\|_p \leqslant \varepsilon$ [$\|x\|_1 \leqslant \varepsilon$].

Corollary 3.1 holds, of course, for the equation

$$x'(t) + \int_0^t b(t, s)\left[x(s) + g(x(s))\right] ds = f(t) \tag{3.2}$$

for suitable b, g. Using the method of §II, we can establish a related result; instead of requiring x_0 and $\|f\|$ to be small, we assume *a priori* that

$$x(t) \to 0 \quad \text{as } t \to \infty. \tag{3.3}$$

Known sufficient conditions for (3.3), involving the signs of b and its partial derivatives, are discussed in §V below.

THEOREM 3.2. *Let b and f satisfy the hypotheses of Theorem* 2.1 ($p = 1$) *with $a \in L^1(\mathbf{R}^+)$ or the hypotheses of Theorem* 2.2. *Let $g \in C(\mathbf{R})$ with $g(x) = o(x)$* ($x \to 0$). *Suppose x is a solution of* (3.2), *and assume* (3.3). *Then $x \in L^1(\mathbf{R}^+)$.*

IV. Proofs. For Theorem 2.1, let $0 < \varepsilon < 1/2\|r\|$ and choose corresponding T and c. Set $y(t) = x(t + T)$, $F(t) = f(t + T)$ ($t \geq 0$) and make a change of variables in (1.1) to obtain

$$\begin{aligned} y'(t) &+ \int_0^t a(t - s)y(s)\, ds \\ &= \int_0^t \left[a(t - s) - b(T + t, T + s)\right] y(s)\, ds + F_1(t), \end{aligned} \tag{4.1}$$

with $y(0) = x(T)$, where

$$|F_1(t)| \leq \Big(\max_{0 \leq \tau \leq T} |x(\tau)|\Big) \int_0^T |b(t + T, s)|\, ds + |F(t)|,$$

so that $F_1 \in L^p(\mathbf{R}^+)$. By (1.2), $y = \varphi + \mathfrak{L}y$, where

$$\varphi(t) = r(t)x(T) + \int_0^t r(t - \tau)F_1(\tau)\, d\tau \in L^p(\mathbf{R}^+),$$

$$\mathfrak{L}y(t) = \int_0^t r(t - \tau)\int_0^\tau \left[a(\tau - s) - b(\tau + T, s + T)\right] y(s)\, ds\, d\tau,$$

so that $\mathfrak{L}: L^p(\mathbf{R}^+) \to L^p(\mathbf{R}^+)$ satisfies

$$\|\mathfrak{L}z\|_p \leq \|r\|\; \|c\|\; \|z\|_p \leq \tfrac{1}{2}\|z\|_p, \tag{4.2}$$

by (1.5). For $0 < \rho < \infty$, let

$$y_\rho(t) = y(t) \quad (0 \leq t \leq \rho), \qquad y_\rho(t) = 0 \quad (\rho < t < \infty).$$

Clearly $y_\rho \in L^p(\mathbf{R}^+)$ and

$$|y_\rho(t)| \leq |\varphi(t)| + |\mathfrak{L}y_\rho(t)| \qquad (0 \leq t < \infty).$$

By Minkowski's inequality and (4.2),

$$\|y_\rho\|_p \leq 2\|\varphi\|_p \qquad (0 < \rho < \infty).$$

It follows that $y \in L^p(\mathbf{R}^+)$ as claimed.

For Theorem 2.2, choose T so large that

$$\|r'\|\,\|1-\gamma(T+\cdot)\|_\infty + \|r\|\,\|A(\cdot)\alpha(T+\cdot)\|\,\|\beta\|_\infty < \tfrac{1}{2}. \tag{4.3}$$

T exists, since $\gamma \to 1$ and for $M, N > 0$

$$\int_0^\infty A(t)\alpha(M+N+t)\,dt \leqslant \alpha(M+N)\int_0^M A(t)\,dt + \int_M^\infty A(t)\alpha(t)\,dt;$$

we obtain (4.3) by choosing first M, then N, sufficiently large, $T = M + N$. Now let $y(t) = x(t+T)$ and use (1.2) as above. These results

$$y = \psi + \mathfrak{L}_1 y + \mathfrak{L}_2 y, \tag{4.4}$$

where

$$\psi(t) = r(t)x(T) + \int_0^t r(t-\tau)\left[f(\tau+T) - \int_0^T b(\tau+T,s)x(s)\,ds\right]d\tau, \tag{4.5}$$

$$\mathfrak{L}_1 y(t) = -\int_0^t r(t-\tau)\int_0^\tau \beta(s+T)\alpha(\tau+T)A(\tau-s)y(s)\,ds\,d\tau, \tag{4.6}$$

$$\mathfrak{L}_2 y(t) = \int_0^t r(t-\tau)\int_0^\tau [1-\gamma(s+T)]a(\tau-s)y(s)\,ds\,d\tau. \tag{4.7}$$

Now

$$\int_0^t r(t-\tau)\int_0^T b(\tau+T,\ s)x(s)\,ds\,d\tau$$
$$= \int_0^T x(s)[\gamma(s)\psi_1(t,s) + \beta(s)\psi_2(t,s)]\,ds. \tag{4.8}$$

Here, by (1.3),

$$\psi_1(t,s) = \int_0^t r(t-\tau)a(\tau+T-s)\,d\tau = \int_{T-s}^{t+T+s} r(t+T-s-\sigma)a(\sigma)\,d\sigma$$
$$= -r'(t+T-s) - \int_0^{T-s} r(t+T-s-\sigma)a(\sigma)\,d\sigma,$$

so for $0 \leqslant s \leqslant T, M > 0$,

$$\int_0^M |\psi_1(t,s)|\,dt \leqslant \|r'\| + \int_0^T a(\sigma)\int_0^{M+T}|r(t)|\,dt\,d\sigma$$
$$\leqslant \|r'\| + \|r\|\int_0^T a(\sigma)\,d\sigma \equiv K < \infty.$$

Similarly,

$$\psi_2(t,s) = \int_0^t r(t-\tau)\alpha(\tau+T)A(\tau+T-s)\,d\tau,$$

$$\int_0^\infty |\psi_2(t,s)|\,dt \leqslant \|r\|\int_0^\infty \alpha(\tau+T)A(\tau+T-s)\,d\tau$$
$$\leqslant \|r\|\,\|\alpha A\| < \infty \qquad (0 \leqslant s \leqslant T).$$

Since $\beta, \gamma \in L^\infty(0, T)$, $x \in C[0, T]$, we deduce from (1.7), (1.10), (4.5), and (4.8) that $\psi \in L^1(\mathbf{R}^+)$.

From (4.6),

$$\mathfrak{L}_1 y(t) = -\int_0^t y(s)\beta(s+T)\int_s^t r(t-\tau)\alpha(\tau+T)A(\tau-s)\,d\tau\,ds$$

$$= -\int_0^t y(s)\beta(s+T)\int_0^{t-s} r(\sigma)\alpha(t-s+T+s-\sigma)A(t-s-\sigma)\,d\sigma\,ds.$$

Thus

$$\|\mathfrak{L}_1 z\| \leqslant \|z\|\,\|\beta\|_\infty\,\|r\|\,\|A(\cdot)\alpha(T+\cdot)\|. \tag{4.9}$$

Similarly, we see from (4.7) that

$$\|\mathfrak{L}_2 z\| \leqslant \|z\|\,\|1-\gamma(T+\cdot)\|_\infty\,\|r'\|.$$

We use this together with (4.3), (4.4), (4.9) and the reasoning of the previous proof to see that $y \in L^1(\mathbf{R}^+)$ with $\|y\| \leqslant 2\|\psi\|$. This proves Theorem 2.2.

We prove Theorem 3.2 under the assumptions of Theorem 2.1 with $p = 1$ and $a \in L^1(\mathbf{R}^+)$; with obvious modifications, the same proof works if instead b satisfies the hypotheses of Theorem 2.2.

Let $\varepsilon = 1/2\|r\|$ and choose corresponding T', c, as for Theorem 2.1. Let $\eta = 1/4\|r\|(\|a\| + \varepsilon)$ and choose $T > T'$ so that $|g(x(t))| \leqslant \eta|x(t)|$ $(t \geqslant T)$; this is possible, since $x(t) \to 0$ $(t \to \infty)$ and $g(x) = o(x)$ $(x \to 0)$. Now let $y(t) = x(t + T)$; as above, we obtain

$$y = \varphi + \mathfrak{L}y + \mathcal{G}y,$$

where $\varphi \in L^1(\mathbf{R}^+)$, (4.2) holds with $p = 1$, and

$$\mathcal{G}z(t) = -\int_0^t r(t-\tau)\int_0^\tau b(\tau+T, s+T)g(z(s))\,ds\,d\tau.$$

Since $|b(\tau+T, s+T)| \leqslant |a(\tau-s)| + c(\tau-s)$, and since $|g(y(s))| \leqslant \eta|y(s)|$ $(s \geqslant 0)$,

$$|\mathcal{G}y(t)| \leqslant \eta\int_0^t |r(t-\tau)|\int_0^t \big(|a(\tau-s)| + c(\tau-s)\big)|y(s)|\,ds\,d\tau. \tag{4.10}$$

Thus if we define y_ρ as in the proof of Theorem 2.1, (4.10) holds with y_ρ in place of y, and

$$|y_\rho(t)| \leqslant |\varphi(t)| + |\mathfrak{L}y_\rho(t)| + |\mathcal{G}y_\rho(t)| \qquad (0 \leqslant t < \infty).$$

Our choice of η implies that $\|\mathcal{G}y_\rho\| \leqslant \|y_\rho\|/4$; together with (4.2) $(p = 1)$, this gives $\|y_\rho\| \leqslant 4\|\varphi\|$ $(0 < \rho < \infty)$. This proves Theorem 3.2.

V. Discussion and examples. Sufficient conditions for

$$r, r' \in L^1(\mathbf{R}^1) \tag{5.1}$$

follow from a variant of the Wiener-Lévy theorem, proved by D. F. Shea and S. Wainger **[8]** and sharpened by G. S. Jordan and R. L. Wheeler **[4]**.

According to [**4**, Theorem 1], (5.1) holds if

$$a = a_1 + a_2, \qquad a_2 \in L^1(\mathbf{R}^+), \tag{5.2}$$

$$a_1 \in LL^1(\mathbf{R}^+) \textit{ and is nonnegative, nonincreasing, and convex on } (0, \infty), \tag{5.3}$$

and

$$\zeta + \hat{a}(\zeta) \neq 0 \quad (\mathrm{Re}\,\zeta \geqslant 0, \zeta \neq 0), \tag{5.4}$$

where $\hat{a}$ is the Laplace transform of a, extended by continuity to $\{\mathrm{Re}\,\zeta = 0, \zeta \neq 0\}$. (Condition (5.4) always holds when $a_2 \equiv 0$, unless a_1 has a special piecewise linear form [**3**].)

Let

$$b(t, s) = \alpha(t)\beta(s)A_1(t - s) + \gamma(s)A_2(t - s), \tag{5.5}$$

with $\alpha, \beta, \gamma \in L^\infty(\mathbf{R}^+)$. (Kernels of this type have been analyzed by T. R. Kiffe [**5**] and J. J. Levin [**6**].) If $\gamma \equiv 1$, $\alpha(t) \to 1$, $\beta(t) \to 1$ $(t \to \infty)$, the hypotheses of Theorem 2.1 hold if $a = A_1 + A_2$ satisfies (5.2), (5.3), and (5.4) with $A_1 \in L^1(\mathbf{R}^+)$ and $A_2 \in L^p(\mathbf{R}^+)$. For Theorem 2.2 it suffices to assume that $\alpha, \beta, \gamma \in L^\infty(\mathbf{R}^+)$, $\gamma(\infty) = 1$, $\alpha(t) \downarrow 1$ $(t \uparrow \infty)$, $A_1 \geqslant 0$, $\alpha A_1 \in L^1$, and that $a = A_2 = a_1$ satisfies (5.3) and (5.4).

For Theorem 3.2, we must know in advance that x exists on $\mathbf{R}^+$ with $x(\infty) = 0$. According to recent results of M. C. Smith [**9**], (see [**5**], [**6**] for earlier versions), this will be true if (with $k(x) = x + g(x)$)

$$\begin{aligned} &f \in L^1(\mathbf{R}^+), k \in C(\mathbf{R}), xk(x) > 0\ (x \neq 0), \\ &|k(x)| \leqslant M[1 + K(x)] \text{ and } K(x) \geqslant -M\ (x \in \mathbf{R}) \\ &\textit{with } M < \infty, K(x) \to \infty\ (|x| \to \infty) \end{aligned} \tag{5.6}$$

(here $K(x) = \int_0^x k(y)\,dy$) and if b and its derivatives satisfy certain sign and growth conditions. For the kernel b of (5.5), either of the following sets of hypotheses ((5.7) or (5.8)), together with (5.6), is sufficient for Theorem 3.2:

$$A_2 \equiv 0, A_1 = a = a_1 \in C^1 \cap L^1(0, \infty) \textit{ and } (5.3) \textit{ holds}, \tag{5.7i}$$

$$\alpha \in C^1(\mathbf{R}^+), \beta \in C(\mathbf{R}^+), \alpha(t) \downarrow 1, 0 \leqslant \beta(t) \uparrow 1\ (t \uparrow \infty), \tag{5.7ii}$$

$$A_1 \textit{ and } A_2 \textit{ belong to } C^1(0, \infty) \textit{ and each satisfies } (5.3), \tag{5.8i}$$

$$\beta \textit{ and } \gamma \textit{ are continuous and nondecreasing on } \mathbf{R}^+, \beta(\infty) = \gamma(\infty) = 1, \tag{5.8ii}$$

$$\alpha \in C^1(\mathbf{R}^+), \alpha(t) \downarrow 0\ (t \uparrow \infty), \alpha A_1 \in L^1(\mathbf{R}^+). \tag{5.8iii}$$

Another kernel for which our results hold (and which was studied in [**6**]) is

$$b(t, s) = a_1(\alpha(t)(t - s)),$$

where $a = a_1 \in L^1(\mathbf{R}^+)$, (5.3) and (5.4) hold, $\alpha \in C(\mathbf{R}^+)$, and $\alpha(t) \downarrow 1$ $(t \uparrow \infty)$. Then by the mean value theorem,

$$0 \leqslant a_1(t-s) - b(t+T, s+T)$$
$$\leqslant -[\alpha(T) - 1](t-s)a_1'(t-s) \equiv c_T(t-s)$$

and $\|c_T\| \to 0$ as $T \to \infty$. Thus b satisfies the hypotheses of Theorem 2.1 with $p = 1$. For Theorem 3.2, certain additional assumptions are again needed to ensure that a solution x exists with $x(\infty) = 0$. See **[6]**, **[9]**.

For an example where $x \notin L^1(\mathbf{R}^+)$ but $x(t) \to 0$ as $t \to \infty$, choose $a(t) = e^{-t}/4$. Then (1.3) reduces to an ordinary differential equation, and $r(t) = e^{-t/2}(1 + \frac{1}{2}t)$. Let $A = a_1$ satisfy (5.3) with $A(\infty) = 0$ and $\int_0^\infty A(t)\,dt = \infty$. Note that for $t \geqslant 1$,

$$q(t) \equiv \int_0^t r(s)A(t-s)\,ds \geqslant \int_0^1 r(s)A(t-s)\,ds > 2A(t)/3.$$

Thus $q > 0$ and $q \notin L^1(\mathbf{R}^+)$, but $q(\infty) = 0$, since $r \in L^1$ and $A(\infty) = 0$. Now let

$$\beta(s) = \beta_0 \qquad (0 \leqslant s \leqslant 1)$$
$$= 0 \qquad (1 < s < \infty),$$

where the positive number β_0 is chosen so that

$$\int_0^1 [a(t) + \beta_0 A(t)]\,dt < \tfrac{1}{2}.$$

Let $b(t,s) = a(t-s) + \beta(s)A(t-s)$, $f \equiv 0$, and let x be the solution of (1.1). Clearly $\frac{1}{2} \leqslant x(t) \leqslant 1$ $(0 \leqslant t \leqslant 1)$, and the change of variables $y(t) = x(t+1)$ gives

$$y'(t) + \int_0^t a(t-s)y(s)\,ds = -\int_0^1 [a(t-s) + \beta_0 A(t-s)]x(s)\,ds$$

$(t \geqslant 0)$, $y(0) = x(1)$. Thus by (1.2),

$$y(t) = r(t)x(1) - \int_0^1 x(s)\int_0^t r(t-\tau)a(\tau+1-s)\,d\tau\,ds$$
$$-\beta_0\int_0^1 x(s)\int_0^t r(t-\tau)A(\tau+1-s)\,d\tau\,ds$$
$$\equiv r(t)x(1) + y_1(t) + y_2(t).$$

But, as in the proof of Theorem 2.2,

$$\int_0^t r(t-\tau)a(\tau+1-s)\,d\tau$$
$$= -r'(t+1-s) - \int_0^{1-s} r(t+1-s-\sigma)a(\sigma)\,d\sigma,$$

so $y_1 \in L^1(\mathbf{R}^+)$ and $y_1(t) \to 0$ $(t \to \infty)$. On the other hand,

$$\int_0^t r(t-\tau)A(\tau+1-s)\,d\tau$$
$$= q(t+1-s) - \int_0^{1-s} r(t+1-s-\sigma)A(\sigma)\,d\sigma$$
$$= q(t+1-s) + O(te^{-t})$$

as $t \to \infty$, uniformly in $0 \leqslant s \leqslant 1$. Therefore $x(t) \to 0$ $(t \to \infty)$ but $x \notin L^1(\mathbf{R}^+)$, as claimed.

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DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061