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SOME REMARKS ON FUNCTIONS OF Λ-BOUNDED VARIATION

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ABSTRACT. It is shown that if a ΛBV function has no external saltus, then its total Λ-variation is independent of its values at points of discontinuity, and a function which is equal to the given function at points of continuity cannot have a lesser total Λ-variation. Necessary and sufficient conditions are determined for one ΛBV space to contain another and for two spaces to be identical.

The notion of Λ-bounded variation (ΛBV) was suggested by certain conditions which appeared in the study of everywhere convergence of Fourier series [1]–[4] and it proved to have further application to the study of convergence and summability of Fourier series [7], [8]. ΛBV was further examined in [6] and [9] and it has recently been shown to be applicable to the study of the Riemann localization principle for double Fourier series [5]. The results of this note were of importance in that application.

We consider real functions defined on an interval I of R^1. Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive real numbers such that $\sum 1/\lambda_n$ diverges. If, for any sequence $\{[a_n, b_n]\}$ of nonoverlapping intervals in I,

$$\sum_{1}^{\infty} |f(a_n) - f(b_n)|/\lambda_n < \infty,$$

we say that $f \in \Lambda BV$. The supremum of such sums is the total Λ-variation of f, V_f. We may show that this is equivalent to requiring that the analogous sums for finite collections be uniformly bounded with least upper bound V_f.

It is easily seen that a function in ΛBV has only simple discontinuities. A function is said to have an *internal saltus* at a point of discontinuity, a, if

$$\liminf_{x \to a} f(x) \leqslant f(a) \leqslant \limsup_{x \to a} f(x).$$

We will show that if a ΛBV function has an internal saltus at each point of discontinuity, then its total Λ-variation is *independent* of its values at points of discontinuity. Further, of all functions continuous at the points of continuity of a given function, those with internal saltus at points of discontinuity have *minimal* total Λ-variation. We will also determine necessary and sufficient

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conditions for one ΛBV space to be contained in another and for two spaces to be identical.

We will now state our results more precisely.

THEOREM 1. *If $f \in \Lambda BV$ with internal saltus at each of its points of discontinuity, then the total Λ-variation of f is independent of the values of f at its points of discontinuity.*

The following is an obvious consequence of Theorem 1, but it is important enough to make explicit.

COROLLARY. *If two functions in ΛBV agree at points of continuity and have an internal saltus at each point of discontinuity, then they have the same total Λ-variation.*

The next theorem, with Theorem 1, implies that, of all functions continuous at the points of continuity of $f \in \Lambda BV$, those with internal saltus at each of their points of discontinuity have minimal total Λ-variation.

THEOREM 2. *If $f \in \Lambda BV$ and $g = f$ at the points of continuity of f, but has an internal saltus at each of its points of discontinuity, then $g \in \Lambda BV$ and $V_f \geqslant V_g$.*

Let $\Gamma = \{\gamma_n\}$ be a nondecreasing sequence of positive real numbers such that $\sum 1/\gamma_n$ diverges and let ΓBV be the associated set of functions of generalized bounded variation. Our final result is the following

THEOREM 3. *For the spaces ΛBV and ΓBV we have*
(i) $\Lambda BV \subseteq \Gamma BV$ *if and only if*

$$\sum_1^n 1/\gamma_k = O\left(\sum_1^n 1/\lambda_k\right),$$

(ii) $\Lambda BV = \Gamma BV$ *if and only if there are real numbers, c and c', such that*

$$0 < c < \left(\sum_1^n 1/\gamma_k\right) \Big/ \left(\sum_1^n 1/\lambda_k\right) < c' < \infty$$

for all n.

The proofs of Theorems 1 and 2 depend on the following result.

LEMMA. *Let $f\colon I \to R^1$ have only simple discontinuities and an internal saltus at each of its points of discontinuity. Let $\{[a_n, b_n]\}$, $n = 1, \ldots, N$, be a collection of nonoverlapping intervals in I. Then, given any $\varepsilon > 0$, there exists a collection of nonoverlapping intervals in I, $\{[\alpha_n, \beta_n]\}$, $n = 1, \ldots, N'$, such that each α_n and β_n is a point of continuity of f and*

$$\sum_1^{N'} |f(\alpha_n) - f(\beta_n)|/\lambda_n \geqslant \sum_1^{N} |f(a_n) - f(b_n)|/\lambda_n - \varepsilon.$$

PROOF. Let $|f(a) - f(b)|/\lambda$ be one term in $\Sigma = \Sigma|f(a_n) - f(b_n)|/\lambda_n$. Suppose b is a point of discontinuity of f. Let

$$d = \liminf_{x \to b} f(x), \quad D = \limsup_{x \to b} f(x),$$

and b' be a point of continuity of f which is arbitrarily close to b. Since

$$\max\{|f(a) - D|, |f(a) - d|\} \geqslant |f(a) - f(b)|,$$

by choosing b' appropriately, we will have, for any given $\eta > 0$,

$$|f(a) - f(b')| + \eta > |f(a) - f(b)|.$$

If b is not an endpoint of another interval of the collection, then b' may be chosen so that if $[a, b]$ is replaced by $[a, b']$, the intervals in the collection are nonoverlapping. If $[b, c]$ is another interval of our collection with associated $\lambda_n = \bar{\lambda}$, then

$$|f(a) - f(c)| \leqslant |f(a) - f(b)| + |f(b) - f(c)|. \tag{$*$}$$

If equality holds in (*), drop $[a, b]$ and $[b, c]$ from the collection of intervals and add $[a, c]$. Associate with the new interval the $\lambda_n = \min\{\lambda, \bar{\lambda}\}$ and shift the indices so that if $\max\{\lambda, \bar{\lambda}\} = \lambda_k$, then $[a_{k+1}, b_{k+1}]$ becomes the kth interval, $[a_{k+2}, b_{k+2}]$ the $(k + 1)$th interval, and so on. The new sum Σ' will then satisfy

$$\Sigma' \geqslant \Sigma.$$

If inequality holds in (*), $[f(a) - f(b)]$ and $[f(b) - f(c)]$ have *opposite signs*. Suppose $f(a) < f(b)$ and $f(b) > f(c)$. Choosing b' so that $f(b')$ approximates D we have

$$|f(a) - f(b')| + \eta \geqslant |f(a) - f(b)|$$

and

$$|f(b') - f(c)| + \eta \geqslant |f(b) - f(c)|.$$

If $f(a) > f(b)$ and $f(b) < f(c)$, choose b' so that $f(b')$ approximates d. In both cases, replace $[a, b]$ and $[b, c]$ by $[a, b']$ and $[b', c]$ in the collection. Proceeding in this manner at each endpoint of an interval in the collection which is a point of discontinuity, we obtain a collection $\{[\alpha_n, \beta_n]\}$ of the type required and such that

$$\sum_1^{N'} |f(\alpha_n) - f(\beta_n)|/\lambda_n + 2\eta \sum_1^N 1/\lambda_n \geqslant \sum_1^N |f(a_n) - f(b_n)|/\lambda_n.$$

Choosing $\eta = \varepsilon/2\Sigma_1^N 1/\lambda_n$, our argument is complete. □

We are now able to demonstrate our main results.

PROOF OF THEOREM 1. Given $\varepsilon > 0$ there is a collection of nonoverlapping intervals $\{[a_n, b_n]\}$, $n = 1, \ldots, N$, such that

$$V_f - \sum_1^N |f(a_n) - f(b_n)|/\lambda_n < \varepsilon/2.$$

From the lemma, we see that there is a collection of nonoverlapping intervals $\{[\alpha_n, \beta_n]\}$, $n = 1, \ldots, N'$, such that f is continuous at each α_n and β_n and

$$\sum_1^{N'} |f(\alpha_n) - f(\beta_n)|/\lambda_n \geqslant \sum_1^{N} |f(a_n) - f(b_n)|/\lambda_n - \varepsilon/2,$$

from which it follows that

$$V_f - \sum_1^{N'} |f(\alpha_n) - f(\beta_n)|/\lambda_n < \varepsilon.$$

We see then that V_f is the supremum of those sums arising from collections of intervals whose endpoints are points of continuity of f. □

PROOF OF THEOREM 2. The points of continuity of g are either points of continuity of f or points at which f has a removable discontinuity, i.e., points at which the limit of $f(x)$ exists but does not equal the value.

Given $\varepsilon > 0$ and any collection of nonoverlapping intervals $\{[a_n, b_n]\}$, $n = 1, \ldots, N$, there is a collection of nonoverlapping intervals $\{[\alpha_n, \beta_n]\}$, $n = 1, \ldots, N'$, such that each α_n and β_n is a point of continuity of g and

$$\sum_1^{N'} |g(\alpha_n) - g(\beta_n)|/\lambda_n \geqslant \sum_1^{N} |g(a_n) - g(b_n)|/\lambda_n - \varepsilon/2.$$

Let $[\alpha, \beta]$ be one of the $[\alpha_n, \beta_n]$ such that β_n is a point of removable discontinuity of f. Choose $\beta' \in (\alpha, \beta)$ such that it is a point of continuity of f and

$$|f(\beta') - g(\beta)| < \varepsilon/4 \sum_1^{N'} 1/\lambda_n.$$

Whenever α_n or β_n is a point of removable discontinuity of f, replace it by an α_n' or β_n' chosen in this manner. Otherwise, let $\alpha_n' = \alpha_n$, $\beta_n' = \beta_n$. Then

$$\begin{aligned} V_f &\geqslant \sum_1^{N'} |f(\alpha_n') - f(\beta_n')|/\lambda_n \\ &\geqslant \sum_1^{N'} |g(\alpha_n) - g(\beta_n)|/\lambda_n - \varepsilon/2 \\ &\geqslant \sum_1^{N} |g(a_n) - g(b_n)|/\lambda_n - \varepsilon \end{aligned}$$

and so $g \in \Lambda\mathrm{BV}$ with $V_f \geqslant V_g$.

PROOF OF THEOREM 3. It is elementary that $\Lambda\mathrm{BV} \subseteq \Gamma\mathrm{BV}$ if and only if $a_n \searrow 0$ and $\sum_1^\infty a_n/\lambda_n < \infty$ imply $\sum_1^\infty a_n/\gamma_n < \infty$.

Suppose there is a $C > 0$ such that

$$\sum_1^{n} 1/\gamma_k < C \sum_1^{n} 1/\lambda_k$$

for all n. If $a_n \searrow 0$, then

$$\begin{aligned}\sum_1^n a_k/\gamma_k &= \sum_1^{n-1}\left(\sum_1^k 1/\gamma_j\right)(a_k - a_{k+1}) + a_n \sum_1^n 1/\gamma_k \\ &\leqslant C\left[\sum_1^{n-1}\left(\sum_1^k 1/\lambda_j\right)(a_k - a_{k+1}) + a_n \sum_1^n 1/\lambda_k\right] \\ &= C\sum_1^n a_k/\lambda_k,\end{aligned}$$

implying that $\Lambda BV \subseteq \Gamma BV$.

Now suppose that $\Sigma_1^n 1/\gamma_k \neq O(\Sigma_1^n 1/\lambda_k)$. Then there is a sequence of integers $n_k \nearrow \infty$ such that

$$\sum_{n_k+1}^{n_{k+1}} 1/\gamma_n \geqslant 2^k \sum_{n_k+1}^{n_{k+1}} 1/\lambda_n, \quad \sum_{n_k+1}^{n_{k+1}} 1/\lambda_n \leqslant \sum_{n_{k+1}+1}^{n_{k+2}} 1/\lambda_n$$

and $n_0 = 0$. Choose

$$a_i = 1\Big/\left(2^k \sum_{n_k+1}^{n_{k+1}} 1/\lambda_n\right) \quad \text{for } i = n_k + 1, \ldots, n_{k+1}.$$

Then $a_i \searrow 0$,

$$\sum_1^{n_{k+1}} a_i/\gamma_i = \sum_{j=0}^{k}\left[\sum_{n_j+1}^{n_{j+1}} 1/\gamma_n\right]\Big/\left[2^j \sum_{n_j+1}^{n_{j+1}} 1/\lambda_n\right] \geqslant k \to \infty$$

and

$$\sum_1^{n_{k+1}} a_i/\lambda_i = \sum_{j=0}^{k} 1/2^j = O(1).$$

Thus $\Lambda BV - \Gamma BV \neq \varnothing$. □

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