PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 74, Number 1, April 1979

ON THE NUMBER OF LATTICE POINTS IN A COMPACT n-DIMENSIONAL POLYHEDRON

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ABSTRACT. An estimate of lattice points, subjected to the orthogonal group $O(n)$, is obtained for a compact n-dimensional polyhedron.

THEOREM. *Suppose P is a compact n-dimensional polyhedron having volume V. Assume additionally that P contains the origin and that no normal line to ∂P is perpendicular to the radial vector. Let G be the orthogonal group $O(n)$, and let L_g be the image of the integral lattice points under g in G. Let $N(x, g)$ be the number of points in L_g which intersect the set xP, and define $R(x, g)$ to be the difference between $N(x, g)$ and the volume of xP. I.e., $R(x, g) = N(x, g) - Vx^n$. Then there is a positive M, such that*

$$\int_G |R(x, g)|\, dg \leqslant M(n, \varepsilon)(\log x)^{2+\delta},$$

where dg is normalized Haar measure on G.

To prove this theorem we shall apply a similar method to the one used in [**1**] and in my paper [**2**]. Again, I would like to thank Professor Burton Randol for his help and encouragement.

PROOF OF THEOREM. First we would like to estimate the Fourier transform of the characteristic function of P. In polar coordinates (r, θ), $(\theta \in S^{n-1})$, let $F(r, \theta) = \int_P e^{2\pi i(r\theta, Y)}dY$.

By the divergence theorem

$$F(r, \theta) = \frac{1}{2\pi ir} \int_{\partial P} e^{2\pi i(r\theta, Y)}(\theta, n(Y))\, dS_Y$$

where $n(Y)$ is the exterior normal to ∂P. Note that $n(Y)$, as a vectorial function of Y is constant on the faces of ∂P. Let us examine the contribution to this integral of a typical face P_{n-1} of ∂P. Now, the contribution from P_{n-1} can be written:

$$\frac{C_1(\theta)}{2\pi ir} \int_{P_{n-1}} e^{2\pi i(r\theta, Y)}\, dS_Y, \quad \text{with } C_1(\theta) = (\theta, n(Y)).$$

Applying the divergence theorem once more, we find that the last integral is itself a sum of terms of the form

Received by the editors December 18, 1977 and, in revised form, April 7, 1978.
AMS (MOS) subject classifications (1970). Primary 52A30.

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0002-9939/79/0000-0172/$02.00

$$\frac{C_1(\theta)C_2(\theta)}{(2\pi ir)^2}\int_{P_{n-2}} e^{2\pi i(r\theta,Y)}\,dS_Y.$$

By applying the divergence theorem $n-1$ times and at each stage examining a typical face of the boundary, we finally conclude that $F(r,\theta)$ is a sum of terms of the form

$$\frac{C(\theta)}{(2\pi ir)^{n-1}}\int_{P_1} e^{2\pi i(r\theta,Y)}\,dS_Y,\quad \text{where } |C(\theta)|\leqslant 1,$$

and P_1 is a line segment in n-space.

Now, let γ be the smallest nonnegative angle which the vector (r,θ) makes with hyperplanes perpendicular to P_1. Then by Lemma 2 of [1], there exists $M_1>0$ such that

$$\left|\int_{P_1} e^{2\pi i(r\theta,Y)}\,dS_Y\right|\leqslant M_1(\log r)^{1+\delta}\left[r\gamma(\log 1/4\gamma)^{1+\delta}\right]^{-1}$$

and so

$$\left|r^{-n+1}\int_{P_1} e^{2\pi i(r\theta,Y)}\,dS_Y\right|\leqslant M_1r^{-n}(\log r)^{1+\delta}\left[\gamma(\log 1/4\gamma)^{1+\delta}\right]^{-1}.$$

This implies that there exists a function $\Phi(\theta)\in L^1(S^{n-1})$ such that $|F(r,\theta)|\leqslant(\log r)^{1+\delta}r^{-n}\Phi(\theta)$. In more detail, $\Phi(\theta)$ is bounded, except in bands about those equators whose polar axes are parallel to the one-dimensional simplexes of ∂P, and is a sum of terms, each of which is singular at the aforementioned equators. Near such an equator, the singularity of the corresponding term is of the type $\gamma^{-1}(\log 1/4\gamma)^{-1-\delta}$, where γ is the arc length from the equator in the perpendicular direction. (That $\Phi(\theta)$ is in $L^1(S^{n-1})$ follows immediately from the estimate on its behavior near the singularities.)

Now let $J(Y)$ be the characteristic function of P. Then $J(Y/x)$ is the characteristic function of xP, and the Fourier transform of $J(Y/x)$ is $x^nF(xr,\theta)$, if we set $Y=(r,\theta)$.

Note that $N(x,g)=\Sigma_N J(g(N/x))$.

Let $\delta(Y)$ be a nonnegative C^∞ function with support in the unit ball and satisfying $\int_{R^n}\delta(Y)dV_Y=1$. Define $\delta_\varepsilon(Y)=\varepsilon^{-n}\delta(Y/\varepsilon)$. Now $\delta_\varepsilon(Y)$ has support in the ball $|Y|\leqslant\varepsilon$ and its integral is also 1.

Next define $J_\varepsilon(x,Y)=\int_{R^n}\delta_\varepsilon(Y-X)J(X/x)dV_X$ and set $N_\varepsilon(x,g)=\Sigma J_\varepsilon(x,g(N))$. By the Poisson summation formula, this last quantity equals $\Sigma\,\hat{\delta}_\varepsilon(g(N))[x^nF(xg(N))]$, since $J_\varepsilon(xY)$ is C^∞ function with compact support.

$$N_\varepsilon(x,g)=Vx^n+\sum{}'\hat{\delta}_\varepsilon(g(N))\left[x^nF(xg(N))\right]$$

where Σ' means summation over all nonzero integral lattice points.

Now assume that the distance of ∂P from the origin is large. As was pointed out in [2], this entails no loss of generality. We then find that for

$\varepsilon > 0$,

$$N_\varepsilon(x - \varepsilon, g) \leqslant N(x, g) \leqslant N_\varepsilon(x + \varepsilon, g).$$

Thus $N_\varepsilon(x - \varepsilon, g) - Vx^n \leqslant R(x, g) \leqslant N_\varepsilon(x + \varepsilon, g) - Vx^n$. By the right-hand side of the last inequality, we find, substituting our previous expression for $N_\varepsilon(x + \varepsilon, g)$ that

$$R(x, g) \leqslant V((x + \varepsilon)^n - x^n) + \sum{}' |\hat{\delta}_\varepsilon(g(N))|(x + \varepsilon)^n \left| F\left((x + \varepsilon)|N|, g\left(\frac{N}{|N|}\right)\right)\right|.$$

Now $|\hat{\delta}_\varepsilon(Y)| \leqslant M_2(1 + \varepsilon|N|)^{-1}$, so by our estimate for $F(r, \theta)$,

$$R(x, g) \leqslant V\big((x + \varepsilon)^n - x^n\big) + \sum{}'(1 + \varepsilon|N|)^{-1}(\log(x + \varepsilon)|N|)^{1+\delta}|N|^{-n}\Phi(g(\theta)).$$

There is a corresponding inequality going the other way, and we easily conclude, assuming ε small, that

$$|R(x, g)| \leqslant M_3\Big[x^{n-1}\varepsilon + \sum{}'(1 + \varepsilon|N|)^{-1}(\log x|N|)^{1+\delta}|N|^{-n}\Phi(g(\theta))\Big].$$

In particular,

$$\int_G |R(x, g)|dg \leqslant M_3\left[x^{n-1}\varepsilon + \sum{}'(1 + \varepsilon|N|)^{-1}(\log x|N|)^{1+\delta}|N|^{-n}\int_G \Phi(g(\theta))dg\right].$$

Now on the right-hand side, the integral over the group is the same as the integral over S^{n-1}, which is finite, since $\Phi(\theta) \in L^1(S^{n-1})$. We conclude that

$$\int_G |R(x, g)|dg \leqslant M_4\Big[x^{n-1}\varepsilon + \sum{}'(1 + \varepsilon|N|)^{-1}(\log x|N|)^{1+\delta}|N|^{-n}\Big].$$

Now set $\varepsilon = x^{1-n}$. Then

$$\sum (\log x|N|)^{1+\delta}(1 + \varepsilon|N|)^{-1}|N|^{-n} = \sum_{|N|<1/\varepsilon} + \sum_{|N|>1/\varepsilon}.$$

These two sums will be estimated by comparing them with integrals

$$\sum_{|N|<1/\varepsilon} \leqslant \int_1^{x^{n-1}} (\log xr)^{1+\delta}(1 + \varepsilon r)^{-1}r^{-n}r^{n-1}\,dr \leqslant \int_1^{x^{n-1}} (\log xr)^{1+\delta}r^{-1}\,dr = O(\log x)^{2+\delta}.$$

$$\sum_{|N|>1/\varepsilon} \leqslant \int_{x^{n-1}}^{\infty} (\log xr)^{1+\delta}(1 + \varepsilon r)^{-1}r^{-n}r^{n-1}\,dr \leqslant \frac{1}{\varepsilon}\int_{x^{n-1}}^{\infty} (\log xr)^{1+\delta}r^{-2}\,dr = O(\log x)^{2+\delta}.$$

So $\Sigma(\log x|N|)^{1+\delta}(1+\varepsilon|N|)^{-1}|N|^{-n} = O(\log x)^{2+\delta}$ which concludes the proof of theorem.

References

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