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# PIXLEY-ROY AND THE SOUSLIN LINE

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ABSTRACT. Necessary and sufficient conditions are given for normality and metricity of the Pixley-Roy space over a subset of the Souslin line.

The purpose of this paper is to answer a question of E. Parker: *for which subsets $X$ of a Souslin* [1] *line $S$ is the Pixley-Roy* [2] *space $P_X R$ over $X$ normal? for which is it metric?*

Without loss of generality, we assume that $S$ is compact, connected, and without nontrivial separable subintervals. Then: $S = \bigcup_{\alpha \in \omega_1} K_\alpha$ where each $K_\alpha$ is a Cantor set and $K_\alpha \subset K_\beta$ for all $\alpha < \beta$. Let

$$D_\alpha = \left(X - \bigcup_{\beta < \alpha} K_\beta\right) \cap \mathrm{cl}\left(\bigcup_{\beta < \alpha} K_\beta \cap X\right).$$

Consider statements:

(A) $\{\alpha \in \omega_1 | D_\alpha \neq \varnothing\}$ *is not stationary in* $\omega_1$.

(B) $P_{(X \cap K_\alpha)} R$ *is metric for all* $\alpha \in \omega_1$.

(C) $P_{(X \cap K_\alpha)} R$ *is normal for all* $\alpha \in \omega_1$.

We prove:

(I) $P_X R$ *is metric if and only if both* (A) *and* (B) *hold.*

(II) $P_X R$ *is normal if and only if both* (A) *and* (C) *hold.*

If $W$ is a subset of a Cantor set $K$, we know the following:

(D) [2] $P_W R$ *is metric if and only if $W$ is countable.*

(E) [Theorem 4 of this paper] $P_W R$ *is normal if and only if $W^n$ is a $Q$-set*[1] *for all $n \in N$.*

(F) [4] *It is consistent with* ZFC *that both there exists a Souslin line and $P_W R$ is normal only if it is also metric.*

(G) [3] *It is consistent with* ZFC *that there exist both a Souslin line and a $W \subset K$ such that $P_W R$ is normal but not metric.*

Using (D) and (E), (I) and (II) become

(I′) $P_X R$ *is metric if and only if* (A) *holds and $X \cap K_\alpha$ is countable for all $\alpha$.*

(II′) $P_X R$ *is normal if and only if* (A) *holds and $(X \cap K_\alpha)^n$ is a $Q$-set for all $n \in N$ and $\alpha \in \omega_1$.*

$P_X R$ is always a Moore space [2]; thus $P_X R$ is a normal nonmetrizable Moore space if and only if (A) and (C) hold but (B) does not. By (F) and (G)

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[1] A space $S$ is a $Q$-set provided every subset of $S$ is a $G_\delta$-set in $S$.


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it is independent of ZFC whether there is an $X$ such that $P_X R$ is a normal nonmetrizable Moore space even if one requires that $X - K_\alpha \neq \varnothing$ for all $\alpha \in \omega_1$.

My reasons for bothering with all of this are:

(1) I had expected $P_X R$ to be metric only if $X$ were countable (and $P_X R$ to be normal only if $X$ were contained in a Cantor set).

(2) I think the following problem is important and I do not know how to solve it. *Suppose that $W$ is a $Q$-set (contained in a Cantor set). Is $W^2$ (or $W^n$) a $Q$-set*? It is certainly consistent with ZFC that there exist a $Q$-set and that the answer be yes for all $Q$-sets $W$. I conjecture that it is also consistent that the answer be no.

In proving Theorems 2 and 3 we do not use the fact that $S$ has no uncountable family of disjoint open intervals; i.e. $S$ could be any linear space with the structure described in paragraph two; i.e. $S$ could be an Aronszajn line.

The Pixley-Roy space $P_X R$ over a space $X$ is the set of all finite subsets of $X$. If $F \in P_X R$ and $U$ is open in $X$ then $\{G \in P_X R | F \subset G \subset U\}$ is a basic open set in $P_X R$. Throughout the paper we assume that $X \subset S$ and $S$, $K_\alpha$, and $D_\alpha$ are as defined in the second paragraph. Conditions (B) and (C) are obviously necessary for (I) and (II) respectively; we begin by proving that (A) is necessary:

THEOREM 1. *If $\{\alpha \in \omega_1 | D_\alpha \neq \varnothing\}$ is stationary in $\omega_1$, then $P_X R$ is not normal.*

PROOF. Using $<$ here for the order in $S$, let

$$L_\alpha = \left\{ x \in D_\alpha | x \in \mathrm{cl}\left\{ y \in X \cap \left( \bigcup_{\beta < \alpha} K_\beta \right) | y < x \right\} \right\}$$

and

$$R_\alpha = \left\{ x \in D_\alpha | x \in \mathrm{cl}\left\{ y \in X \cap \left( \bigcup_{\beta < \alpha} K_\beta \right) | y > x \right\} \right\}.$$

Since $D_\alpha = L_\alpha \cup R_\alpha$, we assume without loss of generality that $\{\alpha \in \omega_1 | L_\alpha \neq \varnothing\}$ is stationary in $\omega_1$.

Let $\mathcal{I}$ be the set of all nontrivial open subintervals of $S$. There is an $S_0 \in \mathcal{I}$ such that, for all $I \in \mathcal{I}$ with $I \subset S_0$, $\{\alpha \in \omega_1 | L_\alpha \cap I \neq \varnothing\}$ is stationary in $\omega_1$. To see this let $\mathcal{I}^*$ be a maximal family of disjoint members of $\mathcal{I}$ such that for each $I \in \mathcal{I}^*$ there is a closed unbounded subset $\Omega_I$ of $\omega_1$ with $L_\alpha \cap I = \varnothing$ for all $\alpha \in \Omega_I$. If there is an $S_0 \in \mathcal{I}$ contained in $S - \bigcup(\mathcal{I}^*)$, then $S_0$ clearly has the desired properties. Otherwise $\bigcup(\mathcal{I}^*)$ is dense in $S$ and hence, since $\mathcal{I}$ is countable, $S - \bigcup(\mathcal{I}^*)$ is separable. So there is a $\beta$ with $(S - \bigcup(\mathcal{I}^*)) \subset K_\beta$. But then $\{\alpha \in \omega_1 | L_\alpha \neq \varnothing\}$ is not stationary since it does not meet the closed unbounded set $\{\alpha > \beta | \alpha \in \bigcap_{I \in \mathcal{I}^*} \Omega_I\}$.

By induction, for each $\alpha \in \omega_1$ choose $\delta_\alpha \in \omega_1$ and $y_\alpha \in L_{\delta_\alpha} \cap S_0$ in such a way that $\delta_\alpha > \sup\{\Delta_\beta | \beta < \alpha\}$. Let $Y = \{y_\alpha | \alpha \in \omega_1\}$, $Z = X - Y$, and $Y^*$ and $Z^*$ be the set of all singletons from $Y$ and $Z$, respectively. Since $Y^*$ and $Z^*$ are closed and disjoint in $P_X R$, assuming that $P_X R$ is normal there are disjoint open sets $U$ and $V$ in $P_X R$ such that $Y^* \subset U$ and $Z^* \subset V$.

For each $\alpha \in \omega_1$, since $y_\alpha \in D_{\delta_\alpha}$ and $\delta_\alpha > \sup\{\delta_\beta | \beta < \alpha\}$, $y_\alpha \notin \mathrm{cl}\{y_\beta | \beta < \alpha\}$. Thus there is a $J_\alpha \in \mathcal{I}$ such that $y_\alpha$ is the left end point of $J_\alpha$ and $J_\alpha \cap \{y_\beta | \beta < \alpha\} = \varnothing$. Since $\{y_\alpha\} \in Y^* \subset U$, $J_\alpha$ can be chosen in such a way that the unordered pair $\{y_\alpha, x\} \in U$ for all $x \in (J_\alpha \cap X)$.

Using the same type of argument used in finding $S_0$, we can find an $S_1 \subset S_0$ with $S_1 \in \mathcal{I}$ such that if $I \subset S_1$ and $I \in \mathcal{I}$, then $I \cap Y \neq \varnothing$.

For each $\alpha \in \omega_1$ choose a maximal family $\mathcal{I}_\alpha$ of disjoint members of $\{J_\beta | \beta > \alpha\}$. In $\omega_1$ choose $\alpha^* > \sup\{\delta_\beta | J_\beta \in \mathcal{I}_\alpha\}$. Observe that if $x \in S_1 \cap L_\gamma$ for some $\gamma > \alpha^*$, then there is a $J \in \mathcal{J}_\alpha$ with $x \in J$. To see this suppose the contrary. Since $x \in D_\gamma$ and $\gamma > \alpha^*$, there is an $I \in \mathcal{I}$ such that $I \subset S_1$, $x$ is the left end point of $I$, and $I \cap \{y_\beta | \beta \leqslant \alpha^*\} = \varnothing$. Since $I \subset S_1$, there is a $\rho \in \omega_1$ with $y_\rho \in I$. Suppose that $\beta < \alpha^*$. If $y_\rho < y_\beta$ in $S$, then $y_\beta \notin J_\rho$ by definition; thus $J_\rho \cap J_\beta = \varnothing$ since $y_\beta$ is the left end point of $J_\beta$. If $y_\rho > y_\beta$ in $S$, then, since $y_\beta \notin I$, $y_\beta < x$; hence, since $x \notin J_\beta$, $J_\rho \cap J_\beta = \varnothing$. Thus $J_\rho \cap J_\beta = \varnothing$ for all $\beta < \alpha^*$. But this contradicts the maximality of $\mathcal{J}_\alpha$.

Choose an unbounded subset $\Gamma$ of $\omega_1$ such that $\alpha < \gamma \in \Gamma$ implies that $\alpha^* < \gamma$; let $\Gamma^*$ be the set of all limits of $\Gamma$ in $\omega_1$. Since $\Gamma^*$ is closed and unbounded and $S_1 \subset S_0$, there is an $x \in S_1 \cap L_\gamma$ for some $\gamma \in \Gamma^*$. Choose $\gamma_1 < \gamma_2 < \ldots$ in $\Gamma$ having $\gamma$ as a limit. By the above paragraph, for each $n \in N$ there is a $\beta_n$ such that $x \in J_{\beta_n}$ and $J_{\beta_n} \in \mathcal{J}_{\gamma_n}$. Since $\gamma$ is the limit of $\{\delta_{\beta_n}\}$, $\{x\} \in Z^* \subset V$. Also $x$ is a limit point in $S$ of $\{y_{\beta_n} | n \in N\}$. So there is an $n$ such that $\{y_{\beta_n}, x\} \in V$. But $\{y_{\beta_n}, x\} \in U$ by the definition of $J_{\beta_n}$. This contradicts $U \cap V = \varnothing$.

THEOREM 2. *If* (A) *and* (B) *hold, then $P_X R$ is metric.*

PROOF. Let $\mathcal{I}$ be the set of all subsets of $X$ of the form $\{X\}$ or $\{x \in X | p < x\}$ or $\{x \in X | x < q\}$ or $\{x \in X | p < x < q\}$ for some $p$ and/or $q$ in $X \cap S$. These sets form a basis for the topology of $X$. Since each $K_\alpha$ is a Cantor set, for each $\alpha$ there is a countable subset $\mathcal{I}_\alpha$ of $\mathcal{I}$ which is an open basis for $(K_\alpha \cap X)$ in $X$. Let $C_\alpha$ be the set of all "end points" ($p$'s and $q$'s in the description above) of members of $\mathcal{I}_\alpha$. For each $\alpha \in \omega_1$, choose $\alpha^* \in \omega_1$ so that $C_\alpha \subset \mathrm{cl} \bigcup_{\beta < \alpha^*} (X \cap K_\beta)$.

By (A), there is a closed unbounded subset $\Gamma$ of $\omega_1$ such that for all $\alpha \in \Gamma$, if $x \in X - \bigcup_{\beta<\alpha} K_\beta$, then $x \notin \mathrm{cl}(X \cap (\bigcup_{\beta<\alpha} K_\beta))$. For each $\alpha \in \Gamma$, let $\Gamma_\alpha = \{\beta \in \omega_1 | \text{if } \alpha < \gamma \in \Gamma, \text{ then } \alpha \leqslant \beta < \gamma\}$. We assume that $\Gamma$ was chosen so that $\beta \in \Gamma_\alpha$ implies that $\beta^* \in \bigcup_{\gamma \leqslant \alpha} \Gamma_\gamma$.

For $\alpha \in \Gamma$, let $X_\alpha = \bigcup_{\beta \in \Gamma_\alpha} (X \cap K_\beta) - \bigcup_{\beta<\alpha} K_\beta$. Index $\{I \in \bigcup_{\beta \in \Gamma_\alpha} \mathcal{I}_\beta | I \cap X_\gamma = \varnothing \text{ for } \gamma < \alpha\} = \{I_{\alpha n} | n \in N\}$. This is an open basis for $X_\alpha$ in $X$.

If $i \in N$ and $\alpha \in \Gamma$, let $J_{ix} = \bigcap \{I_{\alpha n} | n \leqslant i \text{ and } x \in I_{\alpha n}\}$ (one can let $J_{ix} = X$ if $x \notin I_{\alpha n}$ for any $n \leqslant i$). For $F \in P_X R$ define $U_{iF} = \{G \in P_X R | F \subset G \subset \bigcup_{x \in F} J_{ix}\}$; $\{U_{iF} | i \in N\}$ is an open basis for $F$ in $P_X R$. For $i \in N$ define

$$P_i \left\{ F \in P_X R \middle| \begin{array}{l} \text{If } x \in F \cap X_\alpha, \text{ then } x \in I_{\alpha n} \text{ for some } n < i \\ \text{If } x \in F, z \in F, \text{ and } x \neq z, \text{ then } J_{ix} \cap J_{iz} = \varnothing \end{array} \right\}.$$

By (B), $P_{(X \cap K_\alpha)} R$ is metric and hence, by (D), $X \cap K_\alpha$ is countable for all $\alpha \in \omega_1$. Thus we can index $X_\alpha = \{x_{\alpha n} | n \in N\}$. For $i \in N$, define:

$$P_i^* = \left\{ F \in P_i | \text{If } x_{\alpha n} \in F \text{ and } x_{\alpha k} \notin F \text{ and } k < n, \text{ then } x_{\alpha k} \notin \bigcup_{z \in F} J_{iz} \right\}.$$

We prove that *if $F \in P_i^*$ and $G \in P_j^*$ for some $j > i$ and $U_{iF} \cap U_{jG} \neq \varnothing$, then $F \subset G$*. Since for any $G \in P_X R$ there is a $j > i$ with $G \in P_j^*$ and there are at most finitely many $F \subset G$, this proves that $\{U_{iF} | F \in P_i^*\}$ is locally finite for a fixed $i$. The existence of this $\sigma$-locally finite base implies that $P_X R$ is metric and proves Theorem 2.

Suppose on the contrary that there is an $H \in U_{iF} \cap U_{jG}$ and an $x \in F - G$. Since $x \in F \subset H \in U_{jG}$, there is a $y \in G$ such that $x \in J_{jy}$. Since $y \in G \subset H \in U_{iF}$, there is a $z \in F$ such that $y \in J_{iz}$. Since $x$ and $z$ belong to $F \in P_i$, $x \notin J_{iz}$ unless $x = z$.

There are $\alpha$, $\beta$ and $\gamma$ in $\omega_1$ such that $z \in X_\alpha$, $y \in X_\beta$ and $x \in X_\gamma$.

Observe that $\alpha \leqslant \beta \leqslant \gamma$. For suppose $\alpha > \beta$. Since $J_{iz} \subset I_{\alpha n}$ for some $n$ and $I_{\alpha n} \cap X_\beta = \varnothing$ for all $\beta < \alpha$, this contradicts $y \in X_\beta \cap J_{iz}$. Similarly $\beta \leqslant \gamma$.

Suppose $\alpha < \beta$. Then $\alpha < \gamma$ so $x \neq z$. Since $x \notin J_{iz}$ and $y \in J_{iz}$, there is an end point $p$ of some $I_{\alpha n}$ with $p$ between $x$ and $y$ in $S$; by definition $p \in \mathrm{cl}(\bigcup_{\delta \leqslant \alpha^*} X_\delta)$. Since $\{x, y\} \subset J_{jy}$ and since $J_{jy}$ is an interval, $p \in J_{jy}$. But this is a contradiction since $\alpha^* < \beta$ and $J_{jy} \cap (\bigcup_{\delta < \beta} X_\delta) = \varnothing$.

So we must have $\alpha = \beta$. Recall that $J_{iz} = \bigcap \{I_{\alpha n} | n \leqslant i \text{ and } z \in I_{\alpha n}\}$. Thus, since $y \in J_{iz} \cap X_\alpha$ and $j > i$, $J_{jy} \subset J_{iz}$. Since $x \in J_{jy}$, $x \in J_{iz}$. Thus $x = z$.

So $\alpha = \beta = \gamma$ and $x = z$. Since $x = x_{\alpha k}$ and $y = x_{\alpha h}$ for some $h$ and $k$ in $N$ and $x \neq y$, one of $h$ and $k$ is smaller and either $x \notin J_{jy}$ or $y \notin J_{ix}$; but this contradicts $x \in J_{jy}$, $y \in J_{iz}$, and $x = z$.

THEOREM 3. *If* (A) *and* (C) *hold then $P_X R$ is normal.*

PROOF. Assuming (A) we define $\mathcal{I}$, $\mathcal{I}_\alpha$, $C_\alpha$, $\alpha^*$, $\Gamma$, $\Gamma_\alpha$, $X_\alpha$, $I_{\alpha n}$, $J_{ix}$, $U_{iF}$, and $P_i$ exactly as in the proof of Theorem 2.

Now suppose that $Y$ and $Z$ are disjoint closed subsets of $P_X R$; we must find disjoint open sets separating $Y$ and $Z$ and thus prove that $P_X R$ is normal.

For $F \in P_X R$, let $\phi(F) = \{\alpha \in \omega_1 | F \cap X_\alpha \neq \phi\}$. Let $\Delta = \{\langle \phi, J, I \rangle | \exists F \in P_X R \text{ such that } \phi = \phi(F) \text{ and } J = \bigcup_{x \in F} J_{ix}\}$. For $\langle \phi, J, i \rangle \in \Delta$, define

$$P_{\langle\phi,J,i\rangle} = \left\{ F \in P_i | \phi = \phi(F), J = \bigcup_{x \in F} J_{ix} \right\}.$$

Define $Y_{\langle\phi,J,i\rangle} = \{F \in P_{\langle\phi,J,i\rangle} | Z \cap U_{iF} = \varnothing\}$. Then interchanging $Y$ and $Z$ define $Z_{\langle\phi,J,i\rangle}$.

Observe that $Y_{\langle\phi,J,i\rangle}$ and $P_{\langle\phi,J,i\rangle} - Y_{\langle\phi,J,i\rangle}$ are disjoint subsets of $P_{(X \cap K_{\sup \phi})}R$ as are $Z_{\langle\phi,J,i\rangle}$ and $P_{\langle\phi,J,i\rangle} - Z_{\langle\phi,J,i\rangle}$. Also all of these sets are closed in $P_X R$ since any $F$ belonging to any of them has exactly one member in each of the disjoint $\{J_{ix} | x \in F\}$; and for a fixed $\phi$ and $i$, the possibilities for $\{J_{ix} | x \in F\}$ are finite.

Hence, by (C) there is a function $k_{\langle\phi,J,i\rangle} = k$: $P_X R \to N$ such that $U_{k(F)F} \cap U_{k(G)G} = \varnothing$ whenever $F \in Y_{\langle\phi,J,i\rangle}$ and $G \in P_{\langle\phi,J,i\rangle} - Y_{\langle\phi,J,i\rangle}$ or whenever $G \in Z_{\langle\phi,J,i\rangle}$ and $F \in P_{\langle\phi,J,i\rangle} - Z_{\langle\phi,J,i\rangle}$.

There is also a function $i$: $P_X R \to N$ such that, if $\phi(F) = \phi$, $i(F) = i$, and $\bigcup_{x \in F} J_{ix} = J$, then $F \in Y_{\langle\phi,J,i\rangle}$ if $F \in Y$, and $F \in Z_{\langle\phi,J,i\rangle}$ if $F \in Z$. Observe that $\phi$ and $i$ are finite and that for $\theta \subset \phi$ and $n \leqslant i$ there are only finitely many $K$ with $\langle\theta, K, n\rangle \in \Delta$. So we can also define $j$: $P_X R \to N$ such that $j(F) > i(F)$ and for all $n \leqslant i(F)$, $\theta \subset \phi(F)$, $G \subset F$, and $\langle\theta, K, n\rangle \in \Delta$, $j(F) > k_{\langle\theta,K,n\rangle}(G)$.

Claim. *$\bigcup_{F \in Y} U_{j(F)F}$ and $\bigcup_{G \in Z} U_{j(G)G}$ are disjoint open sets separating $Y$ and $Z$.*

Suppose on the contrary that there are $F \in Y$, $G \in Z$, and $H \in U_{j(F)F} \cap U_{j(G)G}$. Without loss of generality we assume that $i = i(F) \leqslant i(G) < j(G) = j$.

Since $i < j$, using the proof for Theorem 2, if $x \in F - G$ and $x \in X_\alpha$, there is a $y_x \in X_\alpha \cap G$ such that $x \in J_{jy_x}$ and $y \in J_{ix}$. Note that $J_{ix} = J_{iy_x}$.

Let $\phi = \phi(F)$ and $J = \bigcup_{x \in F} J_{ix}$. Then $F \in Y_{\langle\phi,J,i\rangle}$ by the definition of $i = i(F)$.

Let $G' = (F \cap G) \cup \{y_x | x \in F - G\}$. Clearly $\phi(F) = \phi(G') \subset \phi(G)$, $J = \bigcup_{y \in G'} J_{iy}$, and $G' \subset G \subset H \subset J$. So $G \in U_{iG'}$ and $G' \in P_{\langle\phi,J,i\rangle} - Y_{\langle\phi,J,i\rangle}$.

Let $H' = F \cup G'$ and $k = k_{\langle\phi,J,i\rangle}$.

Since $k(F) < j(F)$ we have $F \subset H' \subset H \in U_{j(F)F} \subset U_{k(F)F}$ and thus $H' \in U_{k(F)F}$.

Since $\phi \subset \phi(G)$ and $i \leqslant i(G)$, $k(G') < j(G)$. Also $G' \subset H' \in U_{j(G)G'}$ by the definition of $y_x$ and $G'$. So $H' \in U_{k(G')G'}$. But this contradicts $U_{k(F)F} \cap U_{k(G')G'} = \varnothing$ for $F \in Y_{\langle\phi,J,i\rangle}$ and $G' \in P_{\langle\phi,J,i\rangle} - Y_{\langle\phi,J,i\rangle}$.

Theorem 4. *If $W$ is a subset of a Cantor set $K$, then $P_W R$ is normal if and only if $W^n$ is a $Q$-set for all $n \in N$.*

Proof. Let $K = \{f: N \to 2\}$.

If $F \in P_W R$ and $i \in N$, let $J_{iF} = \{f \upharpoonright i | f \in F\}$ and let $U_{iF} = \{G \in P_W R | F \subset G$ and $J_{iG} \subset J_{iF}\}$. Let $W_n = \{F \in P_W R | \, |F| = n\}$ and, for $F \in W_n$, let $F^* \in W^n$ be the natural ordering of $F$: that is $f < g$ in $F$ if there is a

$k$ such taht $f(i) = g(i)$ for all $i < k$ but $f(k) < g(k)$. $P_W R$ is normal if for every pair $Y$ and $Z$ of disjoint closed sets there is an $i$: $P_W R \to N$ such that $U_{i(F)F} \cap U_{i(G)G} = \emptyset$ for all $F \in Y$ and $G \in Z$.

If $S = \langle f_1 \cdots f_n \rangle \in W^n$ and $i \in N$, define $U_{iS} = \{\langle g_1 g_2 \cdots g_n \rangle \in W^n \mid g_k \restriction i = f_k \restriction i$ for all $k \leqslant n\}$. Both $Y$ and $W^n - Y$ are $G_\delta$-sets in $W^n$ if and only if there is a function $i$: $W^n \to N$ such that, if $S \in Y$ and $T \in W^n - Y$, then either $S \notin U_{i(T)T}$ or $T \notin U_{i(S)S}$.

If $S = \langle f_1 \cdots f_n \rangle \in W^n$ there is $S' = \{f \in S\} \in W_m$ for some $m \leqslant n$. Define $W_m^n = \{S \in W^n \mid S' \in W_m\}$ and let $t_S$: $n \to m$ be the unique function such that $f_j$ is the $t_S(j)$th term of $(S')^*$. Let $\mathfrak{T}_m^n = \{t\colon n \to m\}$ and, for $t \in \mathfrak{T}_m^n$, let $W_{mt}^n = \{S \in W_m^n \mid t_S = t\}$. Observe that each $t \in \mathfrak{T}_m^n$ induces a one-to-one correspondence between $W_{mt}^n$ and $W_m$ (taking $S$ to $S'$). Choose $k_S \in N$ such that $f \neq g$ in $S$, then $f \restriction k_S \neq g \restriction k_S$. Observe:

*If $S$ and $T$ belong to $W_{mt}^n$, $i > k_S$, and $j > k_T$, then $(S' \cup T')$ $\in (U_{iS'} \cap U_{jT'})$ if and only if $T \in U_{iS}$ and $S \in U_{jT}$.* (*)

First we prove that, *If $P_W R$ is normal and $Y \subset W^n$, then $Y$ is a $G_\delta$-set in $W^n$.*

Suppose that $m \leqslant n$ and $t \in \mathfrak{T}_m^n$. It is known that $P_W R$ is normal only if it is hereditarily normal. Thus the open subset $\bigcup_{r \geqslant m} W_r$ of $P_W R$ is normal. Since $W_m$ is closed in $\bigcup_{r \leqslant m} W_r$ and discrete in itself, we can find disjoint open sets in $P_W R$ separating $Y^*$ and $W_m - Y^*$. Since $t$ induces a one-to-one correspondence between $W_m$ and $W_{mt}^n$, there is $i$: $W_{mt}^n \to N$ such that

$$U_{i(S)S'} \cap U_{i(T)T'} = \emptyset$$

if $S \in Y \cap W_{mt}^n$ and $T \in W_{mt}^n - Y$. Since if $S \in W^n$, $S$ belongs to $W_{mt}^n$ for exactly one $m$ and $t$, $i$: $W^n \to N$ is well defined; choose $i(S) > k_S$ for all $S \in W^n$.

This $i$ testifies to $Y$ being a $G_\delta$-set for suppose there were an $S = \langle f_1, \ldots, f_n \rangle \in (Y \cap U_{i(T)T})$ and $T = \langle g_1, \ldots, g_n \rangle \in (U_{i(S)S} - Y)$. Assume without loss of generality that $i(S) \leqslant i(T)$. Since $S \in U_{i(T)T}$ and $k_S < i(S) \leqslant i(T)$, $f_j \restriction i(T) = g_j \restriction i(T)$ for all $j \leqslant n$. Thus if $S \in W_{mt}^n$, $T \in W_{mt}^n$ for the same $m$ and $t$. But this contradicts (*) since $U_{i(S)S'} \cap U_{i(T)T'} = \emptyset$.

We now prove that, *If $W^n$ is a $Q$-set for all $n \in N$, then $P_W R$ is normal.*

Suppose that $Y$ and $Z$ are disjoint closed subsets of $P_W R$. For each $m \in N$, let $J_m = \{f \restriction m \mid f \in K\}$ and for $J \subset J_m$ let

$$Y_J = \{F \in P_W R \mid |J| = |F|, J = J_{mF}, \text{and } U_{mF} \cap Z = \emptyset\}.$$

Let $n = |J|$. Since $\{F^* \mid F \in Y_J\}$ and $W^n - \{F^* \mid F \in Y_J\}$ are both $G_\delta$-sets in $W^n$, and, if $G \in (W_n - Y_J)$, then $G^* \in (W^n - \{F^* \mid F \in Y_J\})$, there is a $j_{JY}$ or $j$: $W_n \to N$ such that, if $F \in Y_J$ and $G \in W_n - Y_J$, then either $F^* \notin U_{j(G)G^*}$ or $G^* \notin U_{j(F)F^*}$. Thus, by (*),

$$(F \cup G) \notin U_{j(F)F} \cap U_{j(G)G}.$$

Interchanging $Y$ and $Z$ define $Z_J$ and $j_{JZ}$.

For each $F \in P_W R$ choose $m(F) \in N$ such that for some $J \subset J_{m(F)}$, $F \in Y_J$ if $F \in Y$, $F \in Z_J$ if $F \in Z$, and $F \in (Y_J \cap Z_J)$ if $F \in P_W R - (Y \cup Z)$. Finally define

$$i(F) = \operatorname{lub}\{j_{JY}(G) + j_{JZ}(G) + m(G) | G \subset F \text{ and } J \subset J_m \text{ for some } m \leqslant m(G)\}.$$

This $i$ witnesses a separation of $Y$ and $Z$. For suppose there were $F \in Y$, $G \in Z$, and $H \in (U_{i(F)F} \cap U_{i(G)G})$. Without loss of generality assume $i(F) \leqslant i(G)$.

Since $F \subset H \in U_{i(G)G}$, $J_{i(G)F} \subset J_{i(G)G}$. Since $m(F) \leqslant i(F) \leqslant i(G)$ and $|F| = |J_{m(F)F}|$, $|F| = |J_{i(G)F}|$ and, for each $f \in F$, there is a $g_f \in G$ such that $g_f \restriction i(G) = f \restriction i(G)$. Let $G' = \{g_f | f \in F\}$. Then $|G'| = |F|$, $J_{i(G)F} = J_{i(G)G'}$, and $(G' \cup F) \in (U_{i(F)F} \cap U_{i(G)G'})$. Since $G \subset H \in U_{i(F)F}$, $J_{i(F)G} \subset J_{i(F)F}$ and thus $G \in U_{i(F)G'}$.

*Case* (1) $m(F) \leqslant m(G')$. Let $J = J_{m(F)}$. Then, by the definition of $m$, $F \in Y_J$. However $G' \in (W_{|J|} - Y_J)$ since $m(F) \leqslant i(F)$ and $G \in (U_{i(F)G'} \cap Z)$. Thus, by the definition of $i$, $i(F) \geqslant j_{JY}(F)$ and, since $m(F) \leqslant m(G')$, $i(G) \geqslant j_{JY}(G')$. So, by the definition of $j_{JY}$, $(G' \cup F) \notin (U_{i(F)F} \cap U_{i(G)G'})$ which is a contradiction.

*Case* (2). $m(G') < m(F)$. Since $m(F) \leqslant i(F)$, $G \in (U_{m(F)G'} \cap Z)$; so $(U_{m(G')G'} \cap Z) \neq \varnothing$. Thus, by the definition of $m$, $G' \in Z$. Let $J = J_{m(G')G'}$. Then $G' \in Z_J$ and $F \in (W_{|J|} - Z_J)$. Also $i(G) \geqslant j_{JZ}(G')$ and, since $m(G') < m(F)$, $i(F) \geqslant j_{JZ}(F)$. Again $(G' \cup F) \notin (U_{i(F)F} \cap U_{i(G)G'})$ gives us a contradiction.

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