PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 74, Number 1, April 1979

A CHARACTERIZATION FOR THE PRODUCT OF CLOSED IMAGES OF METRIC SPACES TO BE A k-SPACE

YOSHIO TANAKA

ABSTRACT. We give, under [CH], a necessary and sufficient condition for the product of two closed images of metric spaces to be a k-space.

1. Introduction. In [**14**, Theorem 4.3], we proved the following result. Recall that a space X is said to belong to *class* $\mathfrak{X}'$ if it is the union of countably many closed and locally compact subsets X_n such that $F \subset X$ is closed whenever $F \cap X_n$ is closed for all n.

THEOREM 1.0. *Let X and Y be closed s-images of metric spaces. Then $X \times Y$ is a k-space if and only if one of the following three properties holds*:

(1) *X and Y are metrizable spaces.*

(2) *X or Y is a locally compact, metrizable space.*

(3) *X and Y are spaces of class $\mathfrak{X}'$.*

In that place, we raised the question whether this theorem remains true if "s-images" is weakened to "images".

In this paper, under the continuum hypothesis [CH], we shall give the following affirmative answer to this question.

THEOREM 1.1 [CH]. *Let X and Y be closed images of metric spaces under maps f and g respectively. Then $X \times Y$ is a k-space* (*equivalently*, *$f \times g$ is a quotient map by* [**6**, *Theorem* 1.5]) *if and only if one of the three properties of Theorem* 1.0 *holds.*

Throughout this paper, we shall assume that all spaces are regular T_2, and all maps are continuous surjections.

2. Preliminaries. A space X is *Fréchet* if, whenever $x \in \bar{A}$, then some sequence of points of A converges to x. Obviously, every closed image of a first countable space is Fréchet.

Recall that a space X is *strongly Fréchet* [**10**] (= *countably bi-sequential* in the sense of E. Michael [**7**]) if, whenever $\{F_n; n = 1, 2, \dots\}$ is a decreasing sequence accumulating at $x \in X$, there exist $x_n \in F_n$ such that the sequence $\{x_n; n = 1, 2, \dots\}$ converges to x. Clearly every strongly Fréchet space is Fréchet.

Received by the editors November 15, 1977 and, in revised form, June 8, 1978.
AMS (*MOS*) *subject classifications* (1970). Primary 54B10, 54D50; Secondary 54C10, 54E35.
Key words and phrases. k-spaces, Fréchet spaces, strongly Fréchet spaces, Lindelöf spaces.

0002-9939/79/0000-0179/$02.25

LEMMA 2.1 (cf. [7, THEOREM 9.9]). *Let X be the closed image of a metric space* (*more generally, paracompact space*) *under a map f. If X is strongly Fréchet, then $\partial f^{-1}(x)$ is compact for every $x \in X$.*

Since every Fréchet space is a sequential space, by [13, Lemma 2.1 (A) and Proposition 2.4] and [12, Theorem 2.2], we have

LEMMA 2.2. *Let X be a Fréchet space, and let Y be a metric space. Suppose that $X \times Y$ is a k-space. Then X is strongly Fréchet, or Y is locally compact.*

LEMMA 2.3. *Let X be a Fréchet space, or a k-space each of whose points is a G_δ-set. Let Y be the closed image of a collectionwise normal and Fréchet space Z under a map f. Suppose that $X \times Y$ is a k-space. Then X is strongly Fréchet, or every $\partial f^{-1}(y)$ has property* (P) *below.*

(P) *Every subset of cardinality $2^{\aleph_0}$ in $\partial f^{-1}(y)$ has an accumulation point.*

PROOF. Suppose that there is $y_0 \in Y$ such that $\partial f^{-1}(y_0)$ does not have property (P). Then there is a closed discrete subset $\{x_\alpha;\ \alpha \in A\}$ of $\partial f^{-1}(y_0)$ with $|A| = 2^{\aleph_0}$. Since Z is collectionwise normal, there is a discrete open collection $\{U_\alpha;\ \alpha \in A\}$ in Z with $x_\alpha \in U_\alpha$. Since Z is Fréchet, and $x_\alpha \in \overline{U_\alpha - f^{-1}(y_0)}$ for each $\alpha \in A$, then there is a convergent sequence $\{x_{\alpha i};\ i = 1, 2, \ldots\}$ of $U_\alpha - f^{-1}(y_0)$ with its limit point x_α. Let $C_\alpha = \{x_{\alpha i};\ i = 1, 2, \ldots\} \cup \{x_\alpha\}$ for each $\alpha \in A$, and let $Z_0 = \bigcup_{\alpha \in A} C_\alpha$. Then, since $\{C_\alpha;\ \alpha \in A\}$ is a discrete closed collection in Z, Z_0 is a closed subset of Z. Let $g = f|Z_0$. Then g is a closed map from the locally compact, metric space Z_0. Let $Y_1 = \{y \in Y_0;\ g^{-1}(y)$ is not compact$\}$, where $Y_0 = g(Z_0)$. Then, by [8, Theorem 4], Y_1 is a closed discrete subset of Y_0. It is easy to see that $y_0 \in Y_1$. Since the sequence $g(C_\alpha)$ converges to y_0, and Y_1 is closed and discrete, then each C_α intersects only a finite number of $g^{-1}(y)$, $y \in Y_1$. Hence $C'_\alpha = C_\alpha - g^{-1}(Y_1)$ is infinite, which implies that each sequence C'_α converges to x_α. For each $\alpha \in A$, let $A_\alpha = g(C'_\alpha)$. Then $\mathfrak{A} = \{A_\alpha;\ \alpha \in A\}$ is locally finite, hence point-finite in $Y_0 - Y_1$. For, g is a perfect map on $Z'_0 = Z_0 - g^{-1}(Y_1)$ and $\{C'_\alpha; \alpha \in A\}$ is a discrete collection in Z'_0. Since each A_α is countable, for each $\alpha \in A$, $A(\alpha) = \{\beta \in A;\ A_\alpha \cap A_\beta \neq \varnothing\}$ is at most countable. Then, there is a subset A' of A with cardinality $2^{\aleph_0}$, such that $\mathfrak{A}' = \{A_\alpha;\ \alpha \in A'\}$ is pairwise disjoint. Indeed, let $A = \{\alpha;\ \alpha < 2^{\omega_0}\}$. Then, for each α, there is a pairwise disjoint subcollection $\mathfrak{B}_\alpha$ of $\mathfrak{A}$ such that $|\mathfrak{B}_\alpha| \leqslant |\alpha|$ and $\bigcup_{\beta<\alpha}\mathfrak{B}_\beta \subsetneqq \mathfrak{B}_\alpha$. For, let $\{\mathfrak{B}_\beta;\ \beta < \alpha\}$ be defined for each $\beta < \alpha$. Then we can choose $A_{\alpha'} \in \mathfrak{A}$ with

$$A_{\alpha'} \cap \left(\bigcup_{\beta<\alpha} \{A_\delta; A_\delta \in \mathfrak{B}_\beta\} \right) = \varnothing,$$

for each $A(\delta)$ is at most countable and $|\bigcup_{\beta<\alpha}\mathfrak{B}_\beta| \leqslant |\alpha|$ $(\neq 2^{\aleph_0})$. Let $\mathfrak{B}_\alpha = \{A_{\alpha'}\} \cup \bigcup_{\beta<\alpha}\mathfrak{B}_\beta$. Then $\mathfrak{B}_\alpha$ satisfies the conditions. Hence, $\mathfrak{A}' = \bigcup_{\alpha<2^{\omega_0}}\mathfrak{B}_\alpha$ is a pairwise disjoint subcollection of $\mathfrak{A}$ with cardinality $2^{\aleph_0}$.

Now, let $Z_1 = \bigcup_{\alpha \in A'}\{C'_\alpha \cup \{x_\alpha\}\}$. Let $h = f|Z_1$. Then, since Z_1 is closed in Z, h is a closed map, hence is quotient. Moreover, $h(x_\alpha) = y_0$ for each $\alpha \in A'$ and h is one-to-one on $\bigcup_{\alpha \in A'} C'_\alpha$ by the choice of the index set A'. Here, we may assume that $h|C'_\alpha$ is one-to-one for each $\alpha \in A'$. Thus, $h(Z_1)$ can be shown to be homeomorphic to a quotient space Z_1/F_1 obtained from Z_1 identifying all points of $F_1 = h^{-1}(y_0)$.

On the other hand, $X \times h(Z_1)$ is a closed subset of a k-space $X \times Y$, for $h(Z_1)$ is closed in Y. Hence $X \times h(Z_1)$ is a k-space. This implies that $X \times (Z_1/F_1)$, which is homeomorphic to $X \times h(Z_1)$, is a k-space. Thus, by [**15**, Lemma 2.1(2)], X is strongly Fréchet or $\partial_{Z_1}F_1$ has property (P). However, $\partial_{Z_1}F_1$ contains a closed discrete subset $\{x_\alpha;\ \alpha \in A'\}$ of cardinality $2^{\aleph_0}$. Then it does not have property (P). Therefore X is strongly Fréchet. That completes the proof.

PROPOSITION 2.4 [CH]. *Let X be a Fréchet space, or a k-space each of whose points is a* G_δ*-set. Let Y be the closed image of a first countable, paracompact space under a map f. If* $X \times Y$ *is a k-space, then either X is strongly Fréchet, or* $\partial f^{-1}(y)$ *is locally compact and Lindelöf for every* $y \in Y$.

PROOF. Suppose that X is not strongly Fréchet. Then, without [CH], every $\partial f^{-1}(y)$ is locally compact by [**15**, Theorem 2.2]. Moreover, from Lemma 2.3, every $\partial f^{-1}(y)$ has property (P). Then, under [CH] it is easy to see that every $\partial f^{-1}(y)$ is Lindelöf, for every $\partial f^{-1}(y)$ is paracompact.

3. Proof of Theorem 1.1 and a related result.

PROOF OF THEOREM 1.1. The "if" part is that of Theorem 1.0 stated in §1. So we shall prove the "only if" part.

(i) Suppose that every $\partial f^{-1}(x)$ is Lindelöf: If every $\partial g^{-1}(y)$ is also Lindelöf, as in the proof of [**5**, Corollary 1.2], we may assume that X and Y are closed s-images of metric spaces. Thus, by the "only if" part of Theorem 1.0, the assertion holds. If some $\partial g^{-1}(y_0)$ is not Lindelöf, then X is strongly Fréchet by Proposition 2.4. Thus X is metrizable by Lemma 2.1. On the other hand, Y is not strongly Fréchet by Lemma 2.1, for $\partial g^{-1}(y_0)$ is not compact. Hence X is locally compact by Lemma 2.2.

(ii) Suppose that some $\partial f^{-1}(x_0)$ is not Lindelöf: Then, as above, Y is locally compact and metrizable. That completes the proof.

As for the product of closed images of locally compact metric spaces, we have the following theorem, which is an improvement of [**15**,Proposition 2.6 or 2.7]. The "only if" part follows from the proof of Theorem 1.1. The "if" part follows from Proposition 3.2 below.

THEOREM 3.1 [CH]. *Let* $f_i\colon X_i \to Y_i$ $(i = 1, 2)$ *be closed maps such that each* X_i *is a locally compact metric space (more generally, locally compact, Fréchet and paracompact space). Then* $Y_1 \times Y_2$ *is a k-space if and only if either of the following properties holds*:

(1) *Every* $\partial f_1^{-1}(y_1)$ *is compact, or every* $\partial f_2^{-1}(y_2)$ *is compact.* (*Hence,* Y_1 *or* Y_2 *is locally compact.*)

(2) *Every* $\partial f_i^{-1}(y_i)$ *is Lindelöf for* $i = 1, 2$.

PROPOSITION 3.2. (a) [**4**, *Theorem* 3.2] *Let* Y_1 *be a k-space, and let* Y_2 *be a locally compact space. Then* $Y_1 \times Y_2$ *is a k-space.*

(b) [**15**, *Lemma* 2.5] *Let* Y_i $(i = 1, 2)$ *be closed images of locally compact spaces under maps* f_i *with each* $\partial f_i^{-1}(y_i)$ *Lindelöf. Then* $Y_1 \times Y_2$ *is a k-space.*

4. Some remarks to Theorem 1.1.

REMARK 4.1. Concerning the "Fréchetness" for the product of two closed images of metric spaces, we have the following theorem from [**9**, Theorem 9.2] (also cf. [**7**, Proposition 4.D.5]), together with Lemma 2.1.

THEOREM. *Let* X *and* Y *be closed images of metric spaces. Then* $X \times Y$ *is a Fréchet space* (*equivalently, hereditary k-space by* [**2**]) *if and only if either of the following properties holds*:

(1) X *and* Y *are metrizable spaces.*

(2) X *or* Y *is a discrete space.*

REMARK 4.2. Concerning the "k-ness" for the product of countably many copies of a closed image of a metric space, we have the following theorem from [**13**, Theorem 1.3] and [**7**, Theorem 7.3].

THEOREM. *Let* X *be a closed image of a metric space. Then* X^ω *is a k-space if and only if* X *is a metrizable space.*

REMARK 4.3. As generalizations of metric spaces, J. G. Ceder [**3**] introduced three types of topological spaces which he called M_1, M_2 and M_3-spaces, and observed that $M_1 \Rightarrow M_2 \Rightarrow M_3$. An M_1-space is a regular space having a σ-closure preserving base. That every closed image of a metric space is M_1 was proved by F. Slaughter [**11**]. The following example shows that Theorem 1.1 becomes false if "closed images of metric spaces" is weakened to "M_1-spaces", even if in property (1) of Theorem 1.0 we replace "metrizable spaces" by "first countable spaces".

EXAMPLE. Let X be the Nagata space constructed in Example 9.2 in [**3**] ($X = \{(x, y);\ 0 < x < 1, y \geqslant 0\}$: the topology on X has a base consisting of disks missing the x-axis and sets of the form $U_n(p) = \{p\} \cup \{(x, y);\ |x - p| < 1/n$ and y lies below the graph of $(x - p)^2 + (y - n)^2 = n^2\}$). Obviously X is separable, first countable and not second countable. Hence X is not metrizable. The proof that X is M_1, which is due to J. Nagata, is given in [**3**]. Let C be a closed interval contained in $(0, 1)$. Let Y be a quotient space obtained by identifying all points of C, and let f: $X \to Y$ be the natural quotient map. Since C is compact in X, f is a perfect map. Then $Y \times Y$ is a k-space, for it is the perfect image of a first countable space $X \times X$. To show that Y is M_1, let $\mathfrak{B} = \bigcup_{i=1}^{\infty} \mathfrak{B}_i$ be a σ-closure preserving base for X. We may assume that $\mathfrak{B}_i \subset \mathfrak{B}_{i+1}$ for each i, and that each $\mathfrak{B}_i$ is closed under arbitrary unions. Then, since C is compact in X, $\{f(B);\ B \in \mathfrak{B}$ with $C \subset B$ or $C \cap B = \varnothing\}$ is a σ-closure preserving base for Y.

That Y is not first countable will be shown below, hence neither is Y locally compact by [**3**, Corollary 5.7]. Suppose that Y is first countable. Then the compact, separable metric subset C is of countable character in X (Arhangel′skiĭ [**1**, Definition 3.5]). Then, by [**1**, Lemma 3.2], there is a countable collection $\mathfrak{B}$ of open subsets of X such that, if $c \in C$ and $c \in U$ with U open in X, then $c \in V \subset U$ for some $V \in \mathfrak{B}$. This implies that a subspace $C \times \{y;\ y \geqslant 0\}$ of X is second countable. But this is a contradiction, for the subspace is obviously non-second countable.

To show that Y is not a space of class $\mathfrak{T}'$, suppose not. Then Y is the union of countably many closed and locally compact subsets Y_n such that, $F \subset Y$ is closed whenever $F \cap Y_n$ is closed for each n. We may assume that $Y_n \subset Y_{n+1}$ for each n. Then each compact subset of Y is contained in some Y_n. For any $x \in X$, let $\{V_n;\ n = 1, 2, \ldots\}$ be a decreasing local base at x. Then, for some m, $f(V_m) \subset Y_m$, hence $V_m \subset f^{-1}(Y_m)$. While, since f is perfect, $f^{-1}(Y_m)$ is locally compact. Hence, by [**3**, Corollary 5.7], $f^{-1}(Y_m)$ is metrizable, so is V_m. This implies that X is a locally metrizable space. Then X is metrizable, for it is Lindelöf. But, this is a contradiction to the fact that X is nonmetrizable. Thus Y is not a space of class $\mathfrak{T}'$.

References

1. A. V. Arhangel′skiĭ, *Bicompact sets and the topology of spaces*, Trans. Moscow Math. Soc. **13** (1965), 1–62.

2. ______, *A characterization of very k-spaces*, Czechoslovak Math. J. **18** **(93)** (1968), 392–395.

3. J. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–125.

4. D. E. Cohen, *Spaces with weak topology*, Quart. J. Math. Oxford Ser. (2) **5** (1954), 77–80.

5. E. Michael, *A note on closed maps and compact sets*, Israel J. Math. **2** (1964), 173–176.

6. ______, *Bi-quotient maps and Cartesian products of quotient maps*, Ann. Inst. Fourier (Grenoble) **18** (1968), fasc. 2, 287–302.

7. ______, *A quintuple quotient quest*, General Topology and Appl. **2** (1972), 91–138.

8. K. Morita, *On closed mappings*, Proc. Japan Acad. **32** (1956), 539–543.

9. ______, *Some results on M-spaces*, (Topics in topology, Colloq. Math. Societatis János Bolyai, Vol. 8, Keszthely, Hungary, 1972), North-Holland, Amsterdam, 1974, pp. 489–503.

10. F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology and Appl. **1** (1971), 143–154.

11. F. Slaughter, *The closed image of metrizable space is* M_1, Proc. Amer. Math. Soc. **37** (1973), 309–314.

12. Y. Tanaka, *On quasi-k-spaces*, Proc. Japan Acad. **46** (1970), 1074–1079.

13. ______, *Products of sequential spaces*, Proc. Amer. Math. Soc. **54** (1976), 371–375.

14. ______, *A characterization for the products of k-and-*$\aleph_0$*-spaces and related results*, Proc. Amer. Math. Soc. **59** (1976), 149–155.

15. ______, *On the k-ness for the products of closed images of metric spaces*, General Topology and Appl. **9** (1978), 175–183.

Department of Mathematics, Tokyo Gakugei University, 4-1-1 Nukuikita-machi, Koganei-shi, Tokyo, Japan