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THE STABILITY OF THE EQUATION $f(x + y) = f(x)f(y)$

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ABSTRACT. It is proved that if f is a function from a vector space to the real numbers satisfying

$$|f(x + y) - f(x)f(y)| < \delta$$

for some fixed δ and all x and y in the domain, then f is either bounded or exponential.

In a 1941 paper D. H. Hyers proved that if f: $E \to E'$ is a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leqslant \delta$$

for some $\delta > 0$ and all x, y in E, then there is a unique mapping l: $E \to E'$ such that $\|f(x) - l(x)\| \leqslant \delta$ and $l(x + y) = l(x) + l(y)$ for all x, y in E. Hyers called this the stability of the linear equation

$$f(x + y) = f(x) + f(y).$$

J. Aczél and D. Kölzow have recently communicated to us a problem of E. Lukacs. Namely, does the equation $f(x + y) = f(x)f(y)$ have an anlaogous stability theorem, whereby f is approximated by an exponential function? Our main result is the following.

THEOREM 1. *Let V be a vector space over the rationals Q and let f: $V \to R$ be a real valued function such that*

$$|f(x + y) - f(x)f(y)| \leqslant \delta \textit{ for some } \delta > 0 \textit{ and all } x, y \textit{ in } V. \tag{1}$$

Then either $|f(x)| \leqslant \max(4, 4\delta)$, or there is a Q-linear function l: $V \to R$ such that $f(x) = \exp(l(x))$ for all x in V.

We first prove the result when f is defined on the integers Z and takes nonnegative real values, since this case captures the essence of Theorem 1.

THEOREM 2. *Let f: $Z \to R^+$ take nonnegative values and be such that the functional approximation* (1) *holds for all x, y in Z. Then either $f(x) \leqslant \max(4, 4\delta)$ or $f(x) = a^x$ for some real number $a > 0$.*

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We observe that if $f(x) > \max(4, 4\delta)$ for some integer x, then x can be taken to be nonnegative. If x were negative, then $g(n) = f(-n)$ would define a new function g still satisfying (1) and such that for some positive $-x$, $g(-x) > \max(4, 4\delta)$. In the subsequent Lemmas 3, 4 and the proof of Theorem 2, f will denote a function exceeding $\max(4, 4\delta)$ for some nonnegative integer.

LEMMA 3. *If m, n are positive integers and $f(m) \geqslant 2$, then*

$$|f(mn)/f^n(m) - 1| \leqslant 2\delta/f^2(m). \tag{2}$$

PROOF. We first use induction on n to show that

$$|f(nm) - f^n(m)| \leqslant \big(1 + f(m) + \cdots + f^{n-2}(m)\big)\delta. \tag{3}$$

For $n = 1$, (3) is trivial. Assuming (3) holds for some n and using (1) we get:

$$\begin{aligned}
&|f((n+1)m) - f^{n+1}(m)| \\
&\qquad \leqslant |f((n+1)m) - f(nm)f(m)| + |f(nm)f(m) - f^{n+1}(m)| \\
&\qquad \leqslant \delta + f(m)\big(1 + f(m) + \cdots + f^{n-2}(m)\big)\delta \\
&\qquad = \big(1 + f(m) + \cdots + f^{n-1}(m)\big)\delta, \quad \text{as desired.}
\end{aligned}$$

Division by $f^n(m)$ in (3) and the assumption that $f(m) \geqslant 2$ yields (2).

LEMMA 4. *There exists a positive integer k such that for all integers $x \geqslant k$, $f(x) > \max(4, 4\delta)$.*

PROOF. Using the hypothesis on f we choose a positive integer m such that $f(m) > \max(4, 4\delta)$. An application of (1) or (2) shows that $f(lm)$ increases without bound as l increases through the positive integers.

Notice that $f(x) \neq 0$ for all integers x. Indeed, $|f(m) - f(m-x)f(x)| \leqslant \delta$ together with $f(x) = 0$ would yield $4\delta < \delta$.

Choose l large enough so that for $i = 1, 2, \ldots, m-1$, $f(i)f(lm) > \max(5, 5\delta)$. If $x \geqslant lm$, then $x = tm + i$ where $i = 1, 2, \ldots, m-1$ and $t \geqslant l$. By (1) we have $|f(x) - f(i)f(tm)| \leqslant \delta$. Then

$$f(x) \geqslant f(i)f(tm) - \delta > \max(5, 5\delta) - \delta \geqslant \max(4, 4\delta).$$

We may therefore take $k = lm$.

PROOF OF THEOREM 2. Let k be as in Lemma 3 and suppose x, y, z are positive integers with $x, y \geqslant k$. Then, by Lemmas 1 and 2:

$$|f(xyz)/f^{xz}(y) - 1| \leqslant 2\delta/f^2(y) < 1/2$$

and

$$|f(xyz)/f^{yz}(x) - 1| \leqslant 2\delta/f^2(x) < 1/2.$$

Since $f(xyz)/f^{xz}(y)$ and $f(xyz)/f^{yz}(x)$ are both bounded and bounded away from 0 for all z, their quotients $(f^y(x)/f^x(y))^z$ and $(f^x(y)/f^y(x))^z$ are bounded for all z. Thus

$$f^x(y) = f^y(x) \quad \text{for all } x, y \geqslant k. \tag{4}$$

Let $a = (f(k))^{1/k}$. By (4), $f(x) = a^x$ for all $x \geqslant k$.

Now suppose w is any integer and choose x such that $x \geqslant k$ and $x + w \geqslant k$. Then

$$|f(x+w) - f(x)f(w)| \leqslant \delta,$$

or $|a^{x+w} - a^x f(w)| \leqslant \delta$; so that $a^x|a^w - f(w)| \leqslant \delta$. Letting $x \to \infty$ it follows that $f(w) = a^w$. This completes the proof of Theorem 2.

The next two results generalize Theorem 2.

THEOREM 5. *If f: $Z \to R$ satisfies* (1) (*but is allowed to take negative values*), *then either* $|f(x)| \leqslant \max(4, 4\delta)$ *or there is a real b such that $f(x) = b^x$, for all x in Z.*

PROOF. Since f satisfies (1), so does $|f|$: $Z \to R^+$, due to the inequalities

$$\big|\, |f(x+y)| - |f(x)|\,|f(y)| \,\big| \leqslant |f(x+y) - f(x)f(y)| \leqslant \delta.$$

If $|f|$ exceeds $\max(4, 4\delta)$ at some integer, deduce from Theorem 2 that $|f(x)| = a^x$, for $a = |f(1)| > 0$. By further applying (1) we can show that $f(x) = (f(1))^x$, for all x in Z.

THEOREM 6. *Let f: $Q \to R$ satisfy* (1) *for all rationals x, y and some $\delta > 0$. Then either* $|f(x)| \leqslant \max(4, 4\delta)$, *or $f(x) = e^{\gamma x}$ for some real number γ, and all rational x.*

PROOF. Assume $f \geqslant 0$ for the moment. Also suppose that there is a rational r such that $f(r) > \max(4, 4\delta)$. As in Lemma 4, $f(nr)$ increases without bound when the integer n tends to $+\infty$. Because the rationals nr take integral values infinitely often, f is unbounded over Z. By Theorem 5 $f(m) = a^m$ where $a = f(1) > 0$ and m is any integer.

Let $x = p/q$ be any rational. The function g: $Z \to R$ defined by $g(n) = f(nx)$ is unbounded and satisfies (1). By Theorem 5, $g(n) = b^n$ for some b. In particular: $b = g(1) = f(x)$ and $b^q = g(q) = f(qx) = f(p) = a^p$. This yields $f(x) = a^x$, or $f(x) = e^{\gamma x}$ where $a = e^{\gamma}$.

The case where $f \geqslant 0$ fails and $|f|$ exceeds $\max(4, 4\delta)$ at some rational does not arise when Q is the domain of f. This can be seen by considering the function $|f|$, which still satisfies (1). Then $|f(x)| = a^x$ or $f(x) = \pm a^x$. An application of (1) and the fact every rational x is even will show that the negative sign cannot occur for any x.

LEMMA 7. *Let f: $V \to R$ be as in Theorem* 1. *If $f(x) > \max(4, 4\delta)$ at some x in V, then for any y in V there is a θ in R such that $f(ty) = e^{\theta t}$ for all t in Q.*

PROOF. Assume for the moment $f \geqslant 0$. We first observe that if $u \in V$ and $f(u) > \max(4, 4\delta)$, then $f(tu) = e^{\alpha t}$ for some real $\alpha > 0$ and any rational t. Simply consider g: $Q \to R$ given by $g(t) = f(tu)$. The approximation (1) holds for g and since $g(1) > \max(4, 4\delta)$, g is unbounded as noted in Lemma 4. Thus $g(t) = e^{\alpha t}$, due to Theorem 6. That $\alpha > 0$ follows because $e^{\alpha} > \max(4, 4\delta)$.

For a fixed y in V either $f(ty) = 1$ for all rational t, $f(z) > 1$, or $f(z) < 1$ for some rational multiple $z = ky$.

In the first case take $\gamma = 0$.

Now assume $f(z) > 1$ for some $z = ky$. For the given x such that $f(x) > \max(4, 4\delta)$ and any rational t we have $f(tx) = e^{\alpha t}$ for some real $\alpha > 0$. Then we get from (1):

$$\left|f(tx+z) - e^{\alpha t}f(z)\right| \leqslant \delta. \tag{5}$$

From (5) we get that $f(tx+z)$ is unbounded over all t in Q. Because $f(z) > 1$, $f(tx+z) > e^{\alpha t}$ for all sufficiently large t. Hence there is a rational multiple $u = jx$ of x such that $f(u+z) > f(u) > \max(4, 4\delta)$. Applying our first observation we get $\gamma > \beta > 0$ such that for all rational t:

$$f(t(u+z)) = e^{\gamma t} > e^{\beta t} = f(tu).$$

From (1) we deduce that for all t

$$e^{\beta t}\left|e^{(\gamma-\beta)t} - f(tz)\right| = \left|e^{\gamma t} - e^{\beta t}f(tz)\right| \leqslant \delta.$$

Thus $f(tz)$ is unbounded over all t in Q. From Theorem 5 one sees that $f(tz) = e^{\varepsilon t}$ for some real ε and all t in Q. Since z is a multiple of y, $f(ty) = e^{\theta t}$ for some real θ and all t in Q.

The final case is that $f(z) < 1$. Here (5) still applies, $f(tx+z)$ is unbounded over t in Q and $f(tx+z) < e^{\alpha t}$ for all sufficiently large t. (Note f never vanishes, as in Lemma 4.) Hence there is a vector u in V such that $f(u) > f(u+z) > \max(4, 4\delta)$. Let $w = u+z$, $v = -z$. Then $f(w+v) > f(v) > \max(4, 4\delta)$. As in the case above we get that $f(tv) = e^{\varepsilon t}$ for some ε. Again $f(ty) = e^{\theta t}$ for some θ, since v is a multiple of y.

The case where f is not always positive never arises, for the same reasons as in Theorem 6. Thus Lemma 7 is proved.

Proof of Theorem 1. Suppose f is unbounded over V. Let $u, v \in V$. By Lemma 7 there are reals α, β, γ such that for all t in Q, $f(tu) = e^{\alpha t}$, $f(tv) = e^{\beta t}$, $f(t(u+v)) = e^{\gamma t}$. From (1) we get

$$\left|e^{\gamma t} - e^{\alpha t}e^{\beta t}\right| \leqslant \delta \quad \text{for all } t \text{ in } Q.$$

From the nature of exponentials it follows that $\gamma = \alpha + \beta$. Hence $f(u+v) = f(u)f(v)$. It is then well known that $f(x) = \exp(l(x))$ for some Q-linear function l.

Theorem 8. *Suppose V is a real normed linear space, $\delta > 0$ and $f\colon V \to R$ satisfies* (1). *If f is bounded above on a nonvoid open subset of V, then either $|f(x)| \leqslant \max(4, 4\delta)$ for all x in V or there exists a continuous linear $l\colon V \to R$ such that $f(x) = \exp(l(x))$ for all x in V.*

THEOREM 9. *Suppose* $f: R^n \to R$, $\delta > 0$ *and* (1) *holds for all* x, y *in* R^n. *If* f *is bounded above on a subset of* R^n *of positive Lebesgue measure, then either* $|f(x)| \leqslant \max(4, 4\delta)$ *for all* x *in* R^n, *or there is an* n-*tuple* a *such that* $f(x) = \exp(a \cdot x)$ *for all* x *in* R^n.

These last two results follow easily form Theorem 1 and well-known properties of additive functions.

REMARK 10. In Theorems 1, 2, 5, 8 and 9 $\max(4, 4\delta)$ can be replaced by $\rho = (1 + \sqrt{1 + 4\delta})/2$. To see this, it suffices to show that if $|f(x)| > \rho$ for some x then f is unbounded. In that case $|f(x)| > 1$ and $f(x)^2 - |f(x)| > \delta$. Let $\Delta = \delta/(f(x)^2 - |f(x)|)$. Then $0 < \Delta < 1$ and we have from (3),

$$|f(nx) - f^n(x)| \leqslant \delta\left(1 + |f(x)| + \cdots + f(x)^{n-2}\right).$$

Hence

$$|f(nx)/f(x)^n - 1| \leqslant \left(\delta/f(x)^2\right) \sum_{k=0}^{\infty} |f(x)|^{-k} = \Delta$$

for $n = 1, 2, 3, \ldots$. Since $|f(x)| > 1$, $f(nx)$ is unbounded in n.

Notice that if $f(x) = \rho$ for all x then (1) holds.

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