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DOUBLE COMMUTANTS OF $C_{\cdot 0}$ CONTRACTIONS. II

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ABSTRACT. In [13], it was shown that if T is a $C_{\cdot 0}$ contraction with finite defect indices $\infty > \delta_{T^*} > \delta_T$, then $\{T\}'' = \{\phi(T): \phi \in H^\infty\}$. In this note we shall extend this result to $\infty \geqslant \delta_{T^*} > \delta_T$ and show that $\{T\}''$ and H^∞ is isometric isomorphic, and moreover such an operator is reflexive.

Introduction. In this note we use the notations, introduced in [9], without explanation. For a bounded linear operator T on a separable Hilbert space H, the collection of all subspaces of H invariant for T is denoted by Lat T, and the weakly closed algebra generated by T and I is denoted by A_T. An operator T is called reflexive if every bounded operator A satisfying Lat $A \supseteq$ Lat T belongs to A_T.

When T is a special $C_{\cdot 0}$ contraction, the A_T and $\{T\}''$ were investigated by some mathematicians (for unilateral shift see [2] and [11], for C_0 contraction see [1], [8], [14]).

In place of $C_{\cdot 0}$ contraction T with defect indices $\delta_{T^*} = n$, $\delta_T = m$ (necessarily $n \geqslant m$) we can consider $S(\theta)$ on $H(\theta)$, which is defined by $H(\theta) = H_n^2 \ominus \theta H_m^2$ and $S(\theta)h = P_{H(\theta)}\lambda h(\lambda)$ for h in $H(\theta)$. In this case, for every ϕ in H^∞, $\phi(S(\theta))$ determined by

$$\phi(S(\theta))h = P_{H(\theta)}\phi h \quad \text{for } h \text{ in } H(\theta)$$

belongs to $A_{S(\theta)}$ ([9] and [10]).

If $n = m < \infty$, then $S(\theta)$ is of class C_0. Let $\theta = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ be 2×1 constant matrix valued function. Then $S(\theta)$ formally defined by the above equation is a unilateral shift. In this note we show that if $n > m$, then $S(\theta)$ has several properties in common with a unilateral shift.

Preliminaries. For an $n \times m$ ($\infty \geqslant n > m$) inner function, Nordgren ([5] for $\infty > n > m$) and Sz.Nagy ([6] for $\infty > m$) showed that there are an $n \times m$ normal inner function $N = \operatorname{diag}(\nu_1, \nu_2, \ldots, \nu_m)$ (cf. (1) of [12]), and bounded $n \times n$ matrices Δ, and Δ^a over H^∞ and $m \times m$ matrices Λ and Λ^a over H^∞ satisfying

$$\Delta\theta = N\Lambda, \quad \Delta\Delta^a = \Delta^a\Delta = \eta_1 I_n \quad \text{and} \quad \Lambda\Lambda^a = \Lambda^a\Lambda = \eta_2 I_m$$

for some η_i in H^∞ such that $\eta_i \wedge \nu_m = 1$ ($i = 1, 2$). Setting

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$Xh = P_{H(N)}\Delta h$ for h in $H(\theta)$, and

$Yf = P_{H(\theta)}\eta_2\Delta^a f$ for f in $H(N)$,

it is well known that X: $H(\theta) \to H(N)$ and Y: $H(N) \to H(\theta)$ are injective, and that $XS(\theta) = S(N)X$ and $YS(N) = S(\theta)Y$.

For such X and Y, it is obvious that

$$XY = \eta(S(N)), \quad YX = \eta(S(\theta)), \quad \text{where } \eta = \eta_1\eta_2.$$

Similarly there are injective X' and Y' such that $X'S(\theta) = S(N)X'$, $Y'S(N) = S(\theta)Y'$, $X'Y' = \eta'(S(N))$, $Y'X' = \eta'(S(\theta))$, where η' and $\eta\nu_m$ are relatively prime functions in H^∞.

We can obtain, in virtue of **[6]**, the same results about the hyperinvariant subspaces of $S(\theta)$ with $\infty \geqslant n > m$ as **[12]**. In particular, we have

LEMMA 1. *$\phi(S(\theta))$ is injective, if and only if $\phi \wedge \nu_m = 1$ (see Corollary 2 of* **[12]**).

LEMMA 2. *$\phi(S(\theta))H(\theta)$ is dense in $H(\theta)$, if and only if ϕ is outer (cf. Corollary 1 of* **[12]**).

LEMMA 3. $\{S(N)\}'' = \{\phi(S(N)): \phi \in H^\infty\}$.

PROOF. Since

$$H(N) = \sum_{i=1}^{m} \oplus H(\nu_i) \oplus \sum_{i=m+1}^{n} H^2 \quad \text{and}$$

$$S(N) = \sum_{i=1}^{m} \oplus S(\nu_i) \oplus \sum_{i=m+1}^{n} \oplus S,$$

where S is a unilateral shift on H^2, it is clear that if $A \in \{S(N)\}''$, then $A = \sum_{i=1}^{n} \oplus A_i$ for $A_i \in \{S(\nu_i)\}''$ ($i = 1, 2, \ldots, m$) and $\sum_{i=m+1}^{n} \oplus A_i \in \{\sum_{i=m+1}^{n} S\}''$. From **[10]**, it follows that $A_i = \phi_i(S(\nu_i))$ for $i = 1, 2, \ldots, m$, and from **[2]**, we can deduce that $\sum_{i=m+1}^{n} \oplus A_i = \phi I_{n-m}$. Define a B in $\{S(N)\}'$ by

$$B\sum_{i=1}^{n} \oplus h_i = \sum_{i=1}^{m} \oplus P_i h_{m+1} \oplus \sum_{i=m+1}^{n} \oplus h_i,$$

where P_i is a projection onto ith component of $H(N)$. Then

$$AB\sum_{i=1}^{n} \oplus h_i = \left(\sum_{i=1}^{m} \oplus P_i\phi_i h_{m+1}\right) \oplus \left(\sum_{i=m+1}^{n} \oplus \phi h_i\right)$$

and

$$BA\sum_{i=1}^{n} \oplus h_i = \left(\sum_{i=1}^{m} \oplus P_i\phi h_{m+1}\right) \oplus \left(\sum_{i=m+1}^{n} \oplus \phi h_i\right).$$

Therefore $AB = BA$ implies that $P_i\phi_i h = P_i\phi h$ for every h in H^2. Consequently we have $\phi_i(S(\nu_i)) = \phi(S(\nu_i))$ and hence $A = \phi(S(N))$.

LEMMA 4. *$S(N)$ is reflexive. Moreover if* Lat $A \supseteq$ Lat $S(N)$, *then* $A = \phi(S(N))$ *for some ϕ in H^∞.*

PROOF. Since each component space of $H(N)$ reduces $S(N)$, it also reduces A, that is, A has the form $A = \sum_{i=1}^{n} \oplus A_i$. Since Lat $A_i \supseteq$ Lat S for $i = m+1, m+2, \ldots,$ from **[11]** we have $A_i = \phi_i I$. Now $\nu_{i+1}/\nu_i \in H^\infty$ $(i = 1, 2, \ldots, m-1)$ implies that $H(\nu_1) \subseteq H(\nu_2) \subseteq \cdots \subseteq H(\nu_m) \subseteq H^2$. Therefore, setting $L_{ij} = \{(P_i x \oplus P_j x): x \in H^2\}$, L_{ij} belongs to Lat $S(N)$. If $i, j \geqslant m+1$, then $AL_{ij} \subseteq L_{ij}$ implies $\phi_i = \phi_j$. If $i \leqslant m < j$, then $AL_{ij} \subseteq L_{ij}$ implies that for every x in $H(\nu_i)$ there is a y in H^2 such that $(A_i x \oplus \phi_j x) = (P_i y \oplus y)$. From this it follows that $A_i = \phi_j(S(\nu_i))$ and hence $A = \phi(S(N))$ for some ϕ in H^∞.

REMARK. Lemma 3 is valid for $n = m < \infty$, but Lemma 4 is not generally valid for $n = m < \infty$.

LEMMA 5. $\{S(\theta)\}'' = \{D: \eta(S(\theta))D = \phi(S(\theta))$ *for some* ϕ *in* $H^\infty\}$.

PROOF. For arbitrary D in $\{S(\theta)\}''$ and any B in $\{S(N)\}'$, set $K = XDYB - BXDY$. Then, since YBX belongs to $\{S(\theta)\}'$ and $XY = \eta(S(N))$ belongs to $\{S(N)\}''$, it follows that

$$YK = YXDYB - YBXDY = DYXYB - DYBXY = 0,$$

which implies $K = 0$. Consequently, from Lemma 3, there is a ϕ in H^∞ such that $XDY = \phi(S(N))$. Because

$$\eta(S(\theta))D\eta(S(\theta)) = YXDYX = Y\phi(S(N))X = \eta(S(\theta))\phi(S(\theta)),$$

from Lemma 1, we have $\eta(S(\theta))D = \phi(S(\theta))$. Conversely if $\eta(S(\theta))D = \phi(S(\theta))$, then for every C in $\{S(\theta)\}'$ it follows that

$$\eta(S(\theta))DC = \phi(S(\theta))C = C\phi(S(\theta)) = C\eta(S(\theta))D = \eta(S(\theta))CD.$$

Hence we have $DC = CD$.

LEMMA 6. *If* $XDY = \phi(S(N))$ *and* $X'DY' = \phi'(S(N))$ *for* ϕ, ϕ' *in* H^∞, *then* D *belongs to* $\{S(\theta)\}''$.

PROOF. By the proof of Lemma 5, we have

$$D\eta(S(\theta)) = \phi(S(\theta)) \quad \text{and} \quad D\eta'(S(\theta)) = \phi'(S(\theta)).$$

Consequently, for arbitrary C in $\{S(\theta)\}'$, we have

$$DC\eta(S(\theta)) = D\eta(S(\theta))C = \phi(S(\theta))C = C\phi(S(\theta)) = CD\eta(S(\theta)),$$

and similarly $DC\eta'(S(\theta)) = CD\eta'(S(\theta))$. Since η and η' are relatively prime, the ranges of $\eta(S(\theta))$ and $\eta'(S(\theta))$ span a dense set in $H(\theta)$. Thus we have $DC = CD$.

Main results.

THEOREM 1. *If* $\infty \geqslant n > m$, *then for every* D *in* $\{S(\theta)\}''$ *there is a unique* ϕ *in* H^∞ *such that* $D = \phi(S(\theta))$. *In this case* $\|\phi(S(\theta))\| = \|\phi\|_\infty$.

THEOREM 2. *If* $\infty \geqslant n > m$, *then* $\mathcal{A}_{S(\theta)} = \{D\colon$ Lat $D \supseteq$ Lat $S(\theta)\} = \{S(\theta)\}'' = \{\phi(S(\theta)): \phi \in H^\infty\}$. *In particular*, $S(\theta)$ *is reflexive*.

PROOF OF THEOREM 2. Assume that Theorem 1 is right. Since

$$A_{S(\theta)} \subseteq \{D\colon \operatorname{Lat} D \supseteq \operatorname{Lat} S(\theta)\}$$

and

$$\{S(\theta)\}'' = \{\phi(S(\theta))\colon \phi \in H^\infty\} \subseteq A_{S(\theta)},$$

we must only show that if $\operatorname{Lat} D \supseteq \operatorname{Lat} S(\theta)$, then D belongs to $\{S(\theta)\}''$. $S(\theta)Y = YS(N)$ implies that if L belongs to $\operatorname{Lat} S(N)$, $\overline{YL}$ belongs to $\operatorname{Lat} S(\theta)$. Therefore

$$XDYL \subseteq XD\overline{YL} \subseteq X\overline{YL} \subseteq \overline{XYL} = \overline{\eta(S(N))L} \subseteq L.$$

From Lemma 4, we have $XDY = \phi(S(N))$. And similarly we have $X'DY' = \phi'(S(N))$. Thus by Lemma 6 we can conclude the proof.

PROOF OF THEOREM 1. Let D belong to $\{S(\theta)\}''$. Then from Lemma 5 and Lemma 1 we can assume that $\phi_1(S(\theta))D = \phi_2(S(\theta))$, where ϕ_1 and ϕ_2 are relatively prime functions in H^∞. Thus, from the lifting theorem, there are an $n \times n$ matrix valued bounded function $\Gamma = (\gamma_{ij})$ over H^∞, and an $m \times n$ matrix valued bounded function $\Omega = (\omega_{ij})$ over H^∞ such that

$$\Gamma\theta H_m^2 \subseteq \theta H_m^2, \quad D = P_{H(\theta)}\Gamma | H(\theta), \quad \|D\| = \|\Gamma\|_\infty = \sup_\lambda \|\Gamma(\lambda)\|, \tag{1}$$

and

$$\phi_2 I_n - \phi_1 \Gamma = \theta\Omega. \tag{2}$$

Since θ is inner, there is an $m \times m$ submatrix θ_a of θ such that $\det \theta_a \neq 0$ (cf. [7]). Since τ is a unitary operator on an n-dimensional space then $S(\tau\theta)$ and $S(\theta)$ are unitarily equivalent, we can assume that the determinant of the first $m \times m$ submatrix of θ is not 0. Set $\theta = (\theta_{ij})$ and let $\theta_a = (0_{a(i)j})$ be an $m \times m$ submatrix of θ such that $1 \leqslant a(1) < a(2) < \cdots < a(m)$. For such a submatrix θ_a we fix a natural number $k(a)$ satisfying $k(a) \neq a(i)$ for $i = 1, 2, \ldots, m$. Let $\theta_a' = (\theta_{a(i)j}')$ be the classical adjoint matrix of θ_a. Then by the same technique as the proof of Theorem 1 of [13], from (2), we have

$$-\phi_1\theta_a' \begin{bmatrix} \gamma_{a(1)\,k(a)} \\ \vdots \\ \gamma_{a(m)\,k(a)} \end{bmatrix} = (\det \theta_a) \begin{bmatrix} \omega_{1\,k(a)} \\ \vdots \\ \omega_{m\,k(a)} \end{bmatrix},$$

and hence

$$-\phi_1(\theta_{k(a)1}, \ldots, \theta_{k(a)m})\theta_a' \begin{bmatrix} \gamma_{a(1)\,k(a)} \\ \vdots \\ \gamma_{a(m)\,k(a)} \end{bmatrix} = (\det \theta_a)(\phi_2 - \phi_1\gamma_{k(a)\,k(a)}). \tag{3}$$

From (3), by simple calculation, we have

$$\phi_1 \det \begin{bmatrix} \theta_{a(1)\,1}, \dots, & \theta_{a(1)\,m}, & \gamma_{a(1)\,k(a)} \\ \vdots & \vdots & \vdots \\ \theta_{a(m)\,1}, \dots, & \theta_{a(m)\,m}, & \gamma_{a(m)\,k(a)} \\ \theta_{k(a)\,1}, \dots, & \theta_{k(a)\,m}, & \gamma_{k(a)\,k(a)} \end{bmatrix} = \phi_2 \det \theta_a. \tag{4}$$

This implies that the inner factor of ϕ_1 is a divisor of $\bigwedge_a \det \theta_a$. But $\phi_1 \wedge \nu_m = 1$ implies that $\phi_1 \wedge (\bigwedge_a \det \theta_a) = 1$. Thus ϕ_1 is outer. For a submatrix θ_a satisfying $1 \leqslant a(1) < \cdots < a(m) \leqslant m + 1$, there is a unique $k(a)$ such that $1 \leqslant k(a) \leqslant m + 1$ and $k(a) \neq a(i)$ for $i = 1, 2, \dots, m$. Conversely, for every $1 \leqslant k \leqslant m + 1$, there is a unique θ_a such that $1 \leqslant a(1) < \cdots < a(m) \leqslant m + 1$ and $k(a) = k$. Thus setting $\xi_{k(a)}(\lambda) = \det \theta_a(\lambda)$, (4) implies that for every k: $1 \leqslant k \leqslant m + 1$,

$$|\phi_2(\lambda)|^2 \, |\xi_k(\lambda)|^2 = |\phi_1(\lambda)|^2 \left| \det \begin{bmatrix} \theta_{1\,1}, \dots, & \theta_{1\,m}, & \gamma_{1\,k} \\ \vdots & \vdots & \vdots \\ \theta_{m\,1}, \dots, & \theta_{m\,m}, & \gamma_{m\,k} \\ \theta_{m+1\,1}, \dots, & \theta_{m+1\,m+1}, & \gamma_{m+1\,k} \end{bmatrix} (\lambda) \right|^2 . \tag{5}$$

From (5) it follows that

$$|\phi_2(\lambda)|^2 \sum_{k=1}^{m+1} |\xi_k(\lambda)|^2$$

$$= |\phi_1(\lambda)|^2 \left\| \begin{bmatrix} \gamma_{1\,1}(\lambda), & \gamma_{2\,1}(\lambda), \dots, & \gamma_{m+1\,1}(\lambda) \\ \vdots & \vdots & \vdots \\ \gamma_{1\,m+1}(\lambda), & \gamma_{2\,m+1}(\lambda), \dots, & \gamma_{m+1\,m+1}(\lambda) \end{bmatrix} \cdot \begin{bmatrix} \xi_1(\lambda) \\ \vdots \\ (-1)^m \xi_{m+1}(\lambda) \end{bmatrix} \right\|^2$$

$$\leqslant |\phi_1(\lambda)|^2 \, \|{}^t\, \Gamma_{m+1}(\lambda)\|^2 \left(\sum_{k=1}^{m+1} |\xi_k(\lambda)|^2 \right),$$

where $\Gamma_{m+1}(\lambda)$ is the first submatrix of $\Gamma(\lambda)$ of order $m + 1$, and ${}^t\Gamma_{m+1}(\lambda)$ denotes the transposed matrix of $\Gamma_{m+1}(\lambda)$. Since by the assumption $\xi_{m+1}(\lambda) \neq 0$ a.e., it follows that

$$|\phi_2(\lambda)|^2 \leqslant |\phi_1(\lambda)|^2 \, \|{}^t\, \Gamma_{m+1}(\lambda)\|^2 \leqslant |\phi_1(\lambda)|^2 \, \|\Gamma\|_\infty^2. \tag{6}$$

Thus there is a ϕ in H^∞ such that $\phi_2 = \phi\phi_1$ and $\|\phi\|_\infty \leqslant \|\Gamma\|_\infty = \|D\|$ (cf. [3]). Hence we have $D = \phi(S(\theta))$ (see [13]). Moreover, since $\|D\| \leqslant \|\phi\|_\infty$ is clear, it follows that $\|D\| = \|\phi\|_\infty$.

Assume that $\phi(S(\theta)) = \psi(S(\theta))$ for ϕ and ψ in H^∞. This implies that there is an $m \times n$ matrix valued bounded function $\Omega'(\lambda)$ over H^∞ such that

$$\phi I_n - \psi I_n = \theta\Omega'. \tag{2$'$}$$

By the same way above we can deduce, from (2)$'$, the next relation

$$-\phi(\theta_{k(a)1}, \ldots, \theta_{k(a)m})\theta_a' \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = (\det \theta_a)(\phi - \psi). \tag{3$'$}$$

Since there is a submatrix θ_a such that $\det \theta_a(\lambda) \neq 0$ a.e., we have $\phi(\lambda) = \psi(\lambda)$ a.e.. Thus we can conclude the proof.

Corollaries. From the theorems above we obtain several results.

COROLLARY 1. *$\phi(S(\theta))$ is boundedly invertible, if and only if ϕ is invertible in H^∞.*

PROOF. Suppose $\phi(S(\theta))D = D\phi(S(\theta)) = 1$. Then D belongs to $\{S(\theta)\}''$. Thus $D = \psi(S(\theta))$ for some ψ in H^∞. Since $I = (\phi\psi)(S(\theta))$, we have $1 = \phi\psi$. The converse assertion is obvious.

COROLLARY 2. *$\phi(S(\theta))$ is not compact for every ϕ in H^∞.*

PROOF. If $\phi(S(\theta))$ is compact, then $(\phi\eta)(S(N)) = X\phi(S(\theta))Y$ is compact. In particular, the multiplication by $\phi\eta$ on H^2, i.e. the analytic Toeplitz operator $T_{\phi\eta}$, is compact. But this is impossible (see [2]).

COROLLARY 3.

$$\sigma_p(S(\theta)) = \{z: |z| < 1, \nu_m(z) = 0\}.$$
$$\sigma_r(S(\theta)) = \{z: |z| < 1, \nu_m(z) \neq 0\}.$$
$$\sigma_c(S(\theta)) = \{z: |z| = 1\}.$$

PROOF. First from Lemma 1 $z \in \sigma_p(S(\theta))$, if and only if $\lambda - z$ and $\nu_m(\lambda)$ are not relatively prime, that is, $\nu_m(z) = 0$. Next, from Lemma 2, $z \in \sigma_r(S(\theta))$, if and only if $\nu_m(z) \neq 0$ and $(\lambda - z)$ is not outer, that is, $|z| < 1$ (cf. [4]). Finally, $z \in \rho(S(\theta))$ if and only if $(\lambda - z)$ is invertible. Thus it is clear $\sigma_c(S(\theta)) = \{z: |z| = 1\}$.

REMARK. Let $\theta = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ be a 2×1 matrix valued inner function. Then $\phi(S(\theta))$ is an analytic Toeplitz operator T_ϕ, and $\nu_m = 1$. In this case all corollaries above are well known.

COROLLARY 4. *V is a Banach space isometry of $\{S(\theta)\}''$ onto itself if and only if for $|a| = 1$, $|b| = 1$, $|c| < 1$,*

$$V\phi(S(\theta)) = a\phi\big(b(S(\theta) - c)(1 - \bar{c}S(\theta))^{-1}\big).$$

In particular if $V(1) = 1$, then V is multiplicative.

PROOF. If V is defined by the above equation, then it is clear that

$$V\phi(S(\theta)) = a\left(\phi\left(b\,\frac{\lambda - c}{1 - \bar{c}\lambda}\right)\right)(S(\theta)).$$

Therefore V is a linear mapping on $\{S(\theta)\}''$.

$$\left\| a\left(\phi\left(b\,\frac{\lambda - c}{1 - \bar{c}\lambda}\right)\right)(S(\theta))\right\| = |a|\,\left\|\phi\left(b\,\frac{\lambda - c}{1 - \bar{c}\lambda}\right)\right\|_\infty = \|\phi\|_\infty = \|\phi(S(\theta))\|.$$

Thus V is isometric. Conversely suppose V a Banach space isometry of $\{S(\theta)\}''$ onto itself. Setting $V\phi(S(\theta)) = \phi_V(S(\theta))$, V_0: $\phi \to \phi_V$ is a Banach space isometry on H^∞. Therefore $(V_0\phi)(\lambda) = a(\phi(\mu))(\lambda)$, where μ is a conformal mapping of the open unit disc onto itself (cf. [4]). Consequently V has the form given above. The rest is trivial.

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REFERENCES

1. H. Bercovici, C. Foiaş and B. Sz.-Nagy, *Compléments à l'étude des opérateurs de classe C_0*, Acta Sci. Math. **37** (1975), 313–322.

2. A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1964), 89–102.

3. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.

4. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

5. E. A. Nordgren, *On quasi equivalence of matrices over H^∞*, Acta Sci. Math. **34** (1973), 301–310.

6. B. Sz.-Nagy, *Diagonalization of matrices over H^∞*, Acta Sci. Math. **38** (1976), 223–238.

7. B. Sz.-Nagy and C. Foiaş, *Jordan model for contraction of class $C._0$*, Acta. Sci. Math. **36** (1974), 305–322.

8. ______, *Commutants and bicommutants of operators of class C_0*, Acta Sci. Math. **38** (1976), 311–315.

9. ______, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam; Academiai Kiadó, Budapest, 1970.

10. D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.

11. ______, *Invariant subspaces and unstarred operator algebras*, Pacific. J. Math. **17** (1966), 511–517.

12. M. Uchiyama, *Hyperinvariant subspaces for contractions of class $C._0$*, Hokkaido Math. J. **6** (1977), 260–272.

13. ______, *Double commutants of $C._0$ contractions*, Proc. Amer. Math. Soc. **69** (1978), 283–288.

14. P. Y. Wu, *Commutants of $C_0(N)$ contractions*, Acta Sci. Math. **38** (1976), 193–202.

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