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# APPROXIMATION BY QUOTIENTS OF RATIONAL INNER FUNCTIONS

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ABSTRACT. Let $u$ be a continuous unimodular function on the $n$-dimensional torus $T^n$. It is shown that $u$ can be uniformly approximated by quotients of rational inner functions.

Let $T$ denote the unit circle. Let $u$ be a Lebesgue measurable function on $T$ such that $|u| = 1$ a.e. In [1] Douglas and Rudin proved that: given $\varepsilon > 0$, there exist inner functions $g_1$ and $g_2$ such that $\|u - g_1\bar{g}_2\| < \varepsilon$. ($\|\cdot\|$ indicates the essential-sup norm.) In this paper we establish a continuous analogue of Douglas and Rudin's result on the torus $T^n$. We show that, if $v$ is a continuous unimodular function on $T^n$, then $v$ can be uniformly approximated by quotients of rational inner functions.

Let $I$ and $I_+$ denote, respectively, the set of integers and the set of nonnegative integers. Let $\alpha = (a, b, \ldots, x) \in I^n$. We will use $z^\alpha$ to denote the function defined on $T^n$ by $z^\alpha(\xi_1, \xi_2 \cdots \xi_n) = \xi_1^a \xi_2^b \cdots \xi_n^x$. A finite linear combination of the $z^\alpha$, where the $\alpha$'s are taken from $I_+^n$, will be called a *polynomial*. If $p = \Sigma c(\alpha) z^\alpha$ is a polynomial, we will use the notation $\alpha(p)$ to denote the $n$-tuple $(\alpha_1(p), \ldots, \alpha_n(p))$, where $\alpha_i(p)$ denotes the maximum $i$th component of any $\alpha$ satisfying $c(\alpha) \neq 0$, and we will use $\tilde{p}$ to indicate the polynomial $\Sigma\overline{c(\alpha)}z^{(\alpha(p)-\alpha)}$. Clearly, polynomials are also well defined over the closure of the open unit polydisk $D^n$. A *rational inner function* on $T^n$ is a function of the form $cz^\alpha(\tilde{p}/p)$, where $c$ is a constant with $|c| = 1$, where $\alpha \in I_+^n$, and where $p$ is a polynomial having no zeros on the closure of $D^n$. (Our definition of rational inner function is essentially the same as the one in Rudin's book [3, p. 110].) Finally, let $U$ and $U_0$ denote, respectively, the set of continuous unimodular functions on $T^n$ and the set of continuous unimodular functions on $T^n$ having continuous logarithms. Note that $U$ and $U_0$ are both groups under the usual operation of (pointwise) multiplication of complex-valued functions.

PROPOSITION. *For each $u \in U$, there is an $\alpha \in I^n$ such that $z^\alpha u \in U_0$.*

PROOF. It can be shown that the factor group $U/U_0$ is isomorphic to the first Čech cohomology group $H^1$ of $T^n$, where the coefficients are taken from $I$. (This can be done quickly by applying the Arens-Royden theorem; see [2].)

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It is an exercise in algebraic topology to show that $H^1$ is isomorphic to $I^n$. Since $\{z^\alpha U_0 | \alpha \in I^n\}$ is a free abelian group having $n$ generators, it follows that $U/U_0 = \{z^\alpha U_0 | \alpha \in I^n\}$.

THEOREM. *Let $V$ denote the closure in the topology of uniform convergence of functions in $U$ of the form $g\bar{g}_1$, where $g$ and $g_1$ are rational inner functions. Then $V = U$.*

PROOF. Since $V$ is a subgroup of $U$ and since $z^\alpha \in V$ for every $\alpha \in I^n$, it follows from the proposition above that $U_0 \subseteq V$ implies $V = U$. Suppose $u \in U_0$, then $u = e^{if}$, where $f$ is a real-valued continuous function on $T^n$. Hence, for each positive integer $m$, we have $u = (u_m)^m$, where $u_m = e^{if/m}$. By choosing $m$ sufficiently large, the real part of $u_m$ can be made uniformly close to 1. It follows that, in order to show that $U_0 \subseteq V$, it suffices to prove that every $u \in U_0$ of the form $u = (\bar{v})^2$, where $\operatorname{Re} v \geqslant \frac{1}{2}$, lies in $V$. Let $\varepsilon > 0$ be given. By the Stone-Weierstrass theorem there exists a polynomial $p$ and an $\alpha \in I_+^n$ such that $\|v - \bar{z}^\alpha p\| < \varepsilon/3$, $\|(1/v) - 1/(\bar{z}^\alpha p)\| < \varepsilon/3$, $\|\bar{z}^\alpha p\|^{-1} < 2$, and $\frac{1}{4} < \operatorname{Re} \bar{z}^\alpha p$. It follows that

$$\|\bar{v}^2 - (z^\alpha \bar{p})/(\bar{z}^\alpha p)\| < \|\bar{v}^2 - \bar{v}/(\bar{z}^\alpha p)\| + \|\bar{v}/(\bar{z}^\alpha p) - (z^\alpha \bar{p})/(\bar{z}^\alpha p)\| < \varepsilon.$$

Note that $\bar{p} = \bar{z}^{\alpha(p)}\tilde{p}$ on $T^n$. Hence, $(z^\alpha \bar{p})/(\bar{z}^\alpha p) = z^{2\alpha - \alpha(p)}\tilde{p}/p$. Thus, the proof will be completed if we can show that $p$ has no zeros in the closure of $D^n$. Note that the function $\operatorname{Re} \bar{z}^\alpha p$ is well defined on the closure of $D^n$ and is harmonic in each variable on $D^n$. It follows from the minimum principle for harmonic functions that $\operatorname{Re} \bar{z}^\alpha p \geqslant \frac{1}{4}$ on the closure of $D^n$. In particular $p$ cannot have a zero on the closure of $D^n$.

## REFERENCES

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