PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 74, Number 2, May 1979

ON UNITARY EQUIVALENCE OF REPRESENTATIONS OF C^*-ALGEBRAS

JOEL ANDERSON[1]

ABSTRACT. Assuming the continuum hypothesis, there are inequivalent irreducible representations of $\mathfrak{B}(\mathcal{H})$ that are pointwise equivalent.

The purpose of this note is to present a theorem concerning equivalence of certain representations of separable C^*-subalgebras of $\mathfrak{B}(\mathcal{H})$. It follows as a corollary to the theorem that there are inequivalent representations of $\mathfrak{B}(\mathcal{H})$ that are pointwise equivalent. This answers a question of Dixmier [**3**, 2.12.22].

Throughout, $\mathcal{H}$ shall denote a complex separable infinite dimensional Hilbert space and $\mathfrak{B}(\mathcal{H})$ the bounded linear operators acting on $\mathcal{H}$. A state f on $\mathfrak{B}(\mathcal{H})$ is said to be *diagonalizable* if there is an orthonormal basis $\{x_n: n \in \omega\}$ for $\mathcal{H}$ (ω denotes the natural numbers) and a free ultrafilter $\mathcal{U}$ of subsets of ω such that

$$f(T) = \lim_{\mathcal{U}} (Tx_n, x_n), \qquad T \in \mathfrak{B}(\mathcal{H}). \tag{1}$$

The state f induces (via the G.N.S. construction) a representation $\{\pi_f, \mathcal{H}_f, x_f\}$ where x_f denotes the canonical cyclic vector.

If $\mathcal{A}$ is a unital separable C^*-subalgebra of $\mathfrak{B}(\mathcal{H})$, let $\mathcal{S}(\mathcal{A})$ denote the set of states on $\mathcal{A}$ that are zero on the compact operators in $\mathcal{A}$. The *essential part of the universal representation* of $\mathcal{A}$ is by definition the direct sum of the representations arising from the elements of $\mathcal{S}(\mathcal{A})$.

THEOREM 1. *If $\mathcal{A}$ is a unital separable C^*-subalgebra of $\mathfrak{B}(\mathcal{H})$ and f is a diagonalizable state on $\mathfrak{B}(\mathcal{H})$ with associated representation $\{\pi_f, \mathcal{H}_f, x_f\}$ then, assuming the continuum hypothesis, the restriction of π_f to $\mathcal{A}$ is unitarily equivalent to the direct sum of an uncountable number of copies of the essential part of the universal representation of $\mathcal{A}$.*

It is convenient to present some parts of the proof of Theorem 1 as separate propositions.

PROPOSITION 2. *If $\mathcal{A}$ is a unital separable C^*-subalgebra of $\mathfrak{B}(\mathcal{H})$ and $f \in \mathcal{S}(\mathcal{A})$, then there is an orthonormal sequence $\{y_n: n \in \omega\}$ in $\mathcal{H}$ such that*

$$f(A) = \lim_{n} (Ay_n, y_n) \tag{2}$$

for A in $\mathcal{A}$.

Received by the editors April 21, 1978.
AMS (MOS) subject classifications (1970). Primary 46K10, 46L05.
[1]Partially supported by a grant from the National Science Foundation.

PROOF. This is a weak version of the theorem in [1]. Alternatively, it is not difficult to construct a proof directly using Glimm's theorem [3, 11.2.1].

The proof of Theorem 1 utilizes the principle of transfinite induction. The following proposition is essentially the induction step in the argument.

PROPOSITION 3. *Suppose $\mathcal{Q}$ is a unital separable C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ and f is a diagonalizable state on $\mathcal{B}(\mathcal{H})$ with associated representation $\{\pi_f, \mathcal{H}_f, x_f\}$. If $\mathcal{K}$ is a separable subspace of $\mathcal{H}_f$ and $\{\pi, \mathfrak{M}, y\}$ is a cyclic representation of $\mathcal{Q}$ that is zero on the compact operators in $\mathcal{Q}$, then there is a subspace $\mathcal{K}_1$ of $\mathcal{H}_f$ such that $\mathcal{K}_1$ is orthogonal to $\mathcal{K}$, $\mathcal{K}_1$ reduces $\pi_f(\mathcal{Q})$ and the representation $\pi_f(\cdot)|_{\mathcal{K}_1}$ of $\mathcal{Q}$ is unitarily equivalent to π.*

PROOF. Define a state g on $\mathcal{Q}$ by

$$g(A) = (\pi(A)y, y), \qquad A \in \mathcal{Q}.$$

As g is zero on the compact operators in $\mathcal{Q}$, by Proposition 2, g has the form (2) for some orthonormal sequence $\{y_n: n \in \omega\}$ in $\mathcal{H}$. Since f is diagonalizable, there are an orthonormal basis $\{x_n: n \in \omega\}$ for $\mathcal{H}$ and a free ultrafilter $\mathcal{U}$ on ω that implement f as in (1).

Choose a subsequence of ω by induction as follows. Fix a countable dense subset $\{A_j\}$ of $\mathcal{Q}$. By [2, Corollary 3] π_f is irreducible, so we may choose a countable set $\{T_k\}$ of operators such that $\{\pi_f(T_k)x_f: k \in \omega\}$ is dense in $\mathcal{K}$. Select $\sigma(1)$ in ω so that

$$|(A_1^* T_1 x_1, y_{\sigma(1)})| < 1$$

and

$$|(\pi(A_1)y, y) - (A_1 y_{\sigma(1)}, y_{\sigma(1)})| < 1.$$

If $\sigma(1), \dots, \sigma(n-1)$ have been picked, choose $\sigma(n) > \sigma(n-1)$ such that

$$|(A_j^* T_k x_n, y_{\sigma(n)})| < 1/n \qquad \text{for } 1 \leq j \leq n \text{ and } 1 \leq k \leq n$$

and

$$|(\pi(A_j)y, y) - (A_j y_{\sigma(n)}, y_{\sigma(n)})| < 1/n \qquad \text{for } 1 \leq j \leq n.$$

Define an isometry V on $\mathcal{H}$ by $Vx_n = y_{\sigma(n)}$, $n = 1, 2, \dots,$ and write $x = \pi_f(V)x_f$. If j and k are natural numbers, then

$$\begin{aligned}(\pi_f(T_k)x_f, \pi_f(A_j)x) &= f(V^*A_j^*T_k) = \lim_{\mathcal{U}} (V^*A_j^*T_k x_n, x_n) \\ &= \lim_{\mathcal{U}} (A_j^* T_k x_n, y_{\sigma(n)}) = 0.\end{aligned}$$

Thus, if $\mathcal{K}_1$ denotes the closed subspace spanned by $\{\pi_f(A_j)x: j \in \omega\}$, then $\mathcal{K}_1$ is orthogonal to $\mathcal{K}$ and $\mathcal{K}_1$ reduces $\pi_f(\mathcal{Q})$. Furthermore, if $j \in \omega$, then

$$\begin{aligned}(\pi_f(A_j)x, x) &= f(V^*A_jV) \\ &= \lim_{\mathcal{U}} (A_j y_{\sigma(n)}, y_{\sigma(n)}) = (\pi(A_j)y, y).\end{aligned}$$

Therefore, $(\pi_f(A)x, x) = (\pi(A)y, y)$ for all A in $\mathcal{A}$ and it follows by a standard argument that π and $\pi_f(\cdot)|_{\mathcal{H}_1}$ are unitarily equivalent.

PROOF OF THE THEOREM. As π_f is zero on the compact operators, $\mathcal{H}_f$ has dimension c, the cardinality of the continuum. Also, the separability of $\mathcal{A}$ implies that $\mathcal{S}(\mathcal{A})$ has cardinality c. By the continuum hypothesis there is a well-ordered enumeration $\{\pi_\alpha, \mathfrak{M}_\alpha, y_\alpha\}_{\alpha<\omega_1}$ of the cyclic representations arising from the states in $\mathcal{S}(\mathcal{A})$ such that for each f in $\mathcal{S}(\mathcal{A})$, the associated representation appears in the enumeration an uncountable number of times. (ω_1 denotes the first uncountable cardinal.)

Let us define subspaces $\mathcal{H}_\alpha$ of $\mathcal{H}_f$ by transfinite induction. Suppose that for some ordinal $\alpha < \omega_1$ and all ordinals $\beta < \alpha$, subspaces $\mathcal{H}_\beta$ of $\mathcal{H}_f$ have been chosen such that

(1) $\mathcal{H}_\beta$ is separable;

(2) if $\gamma < \beta$, then $\mathcal{H}_\gamma$ and $\mathcal{H}_\beta$ are orthogonal;

(3) $\mathcal{H}_\beta$ reduces $\pi_f(\mathcal{A})$;

(4) There is a unitary U_β mapping $\mathcal{H}_\beta$ onto $\mathfrak{M}_\beta$ so that $U_\beta\pi_f(A)x = \pi_\beta(A)U_\beta x$ for A in $\mathcal{A}$ and x in $\mathcal{H}_\beta$.

Let $\mathcal{H}$ denote the closed subspace of $\mathcal{H}_f$ generated by $\{\mathcal{H}_\beta : \beta < \alpha\}$. As α is a countable ordinal, $\mathcal{H}$ is separable. Hence Proposition 3 applies (with $\pi = \pi_\alpha$) and there is a subspace $\mathcal{H}_\alpha$ of $\mathcal{H}_f$ with the required properties. The definition is complete.

Write

$$\mathfrak{N} = \sum_{\alpha<\omega_1} \oplus \mathcal{H}_\alpha$$

and for $x = \sum \oplus x_\alpha$ in $\mathfrak{N}$ define

$$Ux = \sum_{\alpha<\omega_1} \oplus U_\alpha x_\alpha.$$

Clearly, $\mathfrak{N}$ reduces $\pi_f(\mathcal{A})$ and, for A in $\mathcal{A}$,

$$U\left(\pi_f(A)|_{\mathfrak{N}}\right)U^* = \sum_{\alpha<\omega_1} \oplus \pi_\alpha(A).$$

If $\mathfrak{N} = \mathcal{H}_f$ we are done. If $\mathfrak{N} \neq \mathcal{H}_f$, then the representation $\pi_f(\cdot)|_{\mathfrak{N}^\perp}$ of $\mathcal{A}$ decomposes into the direct sum of cyclic representations each of which is unitarily equivalent to some π_α. It follows that the restriction of π_f to $\mathcal{A}$ is unitarily equivalent to

$$\left(\sum_{\alpha<\omega_1} \oplus \pi_\alpha\right) \oplus \left(\sum_{g\in\mathcal{T}} \oplus \pi_g\right) \tag{3}$$

where $\mathcal{T}$ consists of (perhaps repeated) elements of $\mathcal{S}(\mathcal{A})$. As $\mathcal{H}_f$ has cardinality c, $\mathcal{T}$ has cardinality at most c and therefore the representation (3) is equivalent to $\sum_{\alpha<\omega_1} \oplus \pi_\alpha$.

If π and π' are representations of a C^*-algebra $\mathcal{A}$, then π and π' are said to be pointwise equivalent if $\pi(A)$ and $\pi'(A)$ are unitarily equivalent for each A in $\mathcal{A}$.

Corollary 4. *Assuming the continuum hypothesis, there are diagonalizable states f and g on $\mathcal{B}(\mathcal{H})$ such that π_f and π_g are pointwise equivalent but not equivalent.*

Proof. Fix an orthonormal basis $\{x_n: n \in \omega\}$ for $\mathcal{H}$ and define a map of the free ultrafilters on ω into the pure states on $\mathcal{B}(\mathcal{H})$ by

$$\mathfrak{U} \mapsto \lim_{\mathfrak{U}} (\cdot x_n, x_n).$$

(By [2, Corollary 3] diagonalizable states are pure.) If $\mathfrak{U}$ and $\mathfrak{V}$ are distinct ultrafilters, then there is a subset σ of ω that is in $\mathfrak{U}$ but not in $\mathfrak{V}$. If P_σ denotes the projection of $\mathcal{H}$ onto the subspace generated by $\{x_n: n \in \sigma\}$ then

$$\lim_{\mathfrak{U}} (P_\sigma x_n, x_n) = 1 \quad \text{and} \quad \lim_{\mathfrak{V}} (P_\sigma x_n, x_n) = 0$$

so that the map is injective. Now there are 2^c free ultrafilters on ω [4], while there are only c unitary operators on $\mathcal{H}$. Therefore [3, 2.8.6] there are diagonalizable states f and g that are not equivalent. On the other hand, by Theorem 1, π_f and π_g are pointwise equivalent.

References

1. J. Anderson, *On vector states and separable C^*-algebras*, Proc. Amer. Math. Soc. **65** (1977), 62–64.

2. ______, *Extreme points in sets of positive linear maps on $\mathcal{B}(\mathcal{H})$*, J. Functional Analysis (to appear).

3. J. Dixmier, *Les C^*-algèbres et leur représentations*, Gauthier-Villars, Paris, 1964.

4. W. Rudin, *Homogeneity problems in the theory of Čech compartifications*, Duke Math. J. **23** (1956), 409–419.

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802