MULTIFUNCTIONS AND CLUSTER SETS

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ABSTRACT. In this article the notion of cluster set–which has been extensively studied within the framework of real and analytic function theory and to some extent for arbitrary functions between arbitrary topological spaces–is investigated for multifunctions. We generalize the notion of cluster set, extend and generalize some results for cluster sets of functions, and offer some results for multifunctions which are new for functions. In the last section, several compactness generalizations are characterized in terms of multifunctions and cluster sets.

1. Introduction. The notion of cluster set has received a great deal of study within the framework of real and analytic function theory (see [1]). In [17], Weston initiated the study of cluster sets for arbitrary functions between topological spaces, and Hunter, in [10], offered an abstract formulation of some theorems on cluster sets. More recently, cluster sets for functions in general topology have been investigated by Hamlett in [4], [5] and [6]. In these papers, interesting results have been reached which relate properties of cluster sets of a function to the continuity properties of the function and to the properties of the graph of the function.

A *multifunction from a set X to a set Y* is a function from X to $\mathcal{P}(Y) - \{\varnothing\}$, where $\mathcal{P}(Y)$ is the power set of Y. Smithson [13] has contributed a survey relating some of the principal results for multifunctions.

In this paper we extend the notion of cluster set to multifunctions, generalize the concept of cluster set, extend and generalize some earlier results for cluster sets of functions and present some results for multifunctions which are also new for functions. We also give new characterizations of compactness, m-compactness, Lindelöfness and H-closeness in terms of cluster sets.

2. Some preliminaries. We will denote the closure of a subset A of a space by $\mathrm{cl}(A)$, the collection of open neighborhoods of A by $\Sigma(A)$, and the collection of closed neighborhoods of A by $\Gamma(A)$. Following Veličko [16], we say that a point x in a space is in the *θ-closure of a subset A* of the space $(x \in \mathrm{cl}_\theta(A))$ if each $V \in \Gamma(x)$ satisfies $A \cap V \neq \varnothing$, that A is *θ-closed* if $\mathrm{cl}_\theta(A) = A$, and that x is in the *θ-adherence of a filterbase* Ω on the space $(x \in \mathrm{ad}_\theta\Omega)$ if $x \in \mathrm{cl}_\theta(F)$ for each $F \in \Omega$. We extend the notions of adherence, ad, and θ-adherence, writing $\mathrm{ad}\,\Omega = \bigcap_\Omega \mathrm{cl}(F)$ and $\mathrm{ad}_\theta\Omega = \bigcap_\Omega \mathrm{cl}_\theta(F)$ for any family, Ω, of subsets.

We will represent the class of all multifunctions from a set X to a set Y by $\mathfrak{M}(X, Y)$. Let X and Y be spaces and let $\alpha \in \mathfrak{M}(X, Y)$. We say that α is

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upper-semicontinuous at $x \in X$ if for each $W \in \Sigma(\alpha(x))$ in Y there is a $V \in \Sigma(x)$ in X satisfying $\alpha(V) \subset W$; α is *upper-semicontinuous (u.s.c.) on* X if α is u.s.c. at each $x \in X$. Smithson [14] has defined α to be *weakly upper-semicontinuous (w.u.s.c.) at* $x \in X$ if for each $W \in \Gamma(\alpha(x))$ there is a $V \in \Sigma(x)$ with $\alpha(V) \subset W$ and has called α *weakly upper-semicontinuous (w.u.s.c.) on* X when α is w.u.s.c. at each $x \in X$. We say that α has *closed (θ-closed) point images* if $\alpha(x)$ is closed (θ-closed) for each $x \in X$. As usual, we will denote the graph of α–i.e. $\{(x, y): x \in X \text{ and } y \in \alpha(x)\}$–by $\mathcal{G}(\alpha)$ and say that α has a *closed graph* if $\mathcal{G}(\alpha)$ is a closed subset of $X \times Y$. Employing the notion of function with a strongly-closed graph from [8] we say that α has a *strongly-closed graph* if for each $(x, y) \in (X \times Y) - \mathcal{G}(\alpha)$ there are sets $V \in \Sigma(x)$, $W \in \Gamma(y)$, in X and Y, respectively, with $(V \times W) \cap \mathcal{G}(\alpha) = \varnothing$. We say that α has a *θ-closed graph* if $\mathcal{G}(\alpha)$ is a θ-closed subset of $X \times Y$. In a product space $X \times Y$ the *(2)θ-closure of a subset* K *($(2)\text{cl}_\theta(K)$) of* $X \times Y$ is $\{(x, y) \in X \times Y: \text{all } V \in \Sigma(x) \text{ and } W \in \Gamma(y) \text{ satisfy } (V \times W) \cap K \neq \varnothing\}$; K is *(2)θ-closed* if $(2)\text{cl}_\theta(K) = K$. Evidently, $\alpha \in \mathfrak{M}(X, Y)$ has a strongly-closed graph if and only if $\mathcal{G}(\alpha)$ is a (2)θ-closed subset of $X \times Y$. If $A \subset X$, $B \subset Y$, $\Delta \subset \mathcal{P}(X)$, $\nabla \subset \mathcal{P}(Y)$, then $\alpha(A) = \bigcup_{x \in A} \alpha(x)$, $\alpha^{-1}(B) = \{x \in X: \alpha(x) \cap B \neq \varnothing\}$, $\alpha(\Delta) = \{\alpha(A): A \in \Delta\}$ and $\alpha^{-1}(\nabla) = \{\alpha^{-1}(B): B \in \nabla\}$.

3. Multifunctions, cluster sets, graph and continuity properties. Let X and Y be spaces and let $\alpha \in \mathfrak{M}(X, Y)$. Utilizing Weston's definition [17] as a model we define the *cluster set of α at* $x \in X$ to be $\mathcal{C}(\alpha; x) = \text{ad } \alpha(\Sigma(x))$. Generalizing this notion we define the *strong cluster set of α at* $x \in X$ to be $\mathcal{S}(\alpha; x) = \text{ad}_\theta \, \alpha(\Sigma(x))$ and the *θ-cluster set of α at* $x \in X$ to be $\mathcal{T}(\alpha; x) = \text{ad}_\theta \, \alpha(\Gamma(x))$. In Lemma 3.1 we relate $\mathcal{C}(\alpha; x)$, $\mathcal{S}(\alpha; x)$ and $\mathcal{T}(\alpha; x)$ to $\alpha(x)$. We prove only statement (a) as the proofs of the other statements are similar; the projection from $X \times Y$ onto Y is represented by π_y.

3.1. LEMMA. *Let X, Y be spaces and let $\alpha \in \mathfrak{M}(X, Y)$. Then*
(a) $\mathcal{C}(\alpha; x) = \pi_y((\{x\} \times Y) \cap \text{cl}(\mathcal{G}(\alpha)))$ *for each* $x \in X$.
(b) $\mathcal{S}(\alpha; x) = \pi_y((\{x\} \times Y) \cap (2)\text{cl}_\theta(\mathcal{G}(\alpha)))$ *for each* $x \in X$.
(c) $\mathcal{T}(\alpha; x) = \pi_y((\{x\} \times Y) \cap \text{cl}_\theta(\mathcal{G}(\alpha)))$ *for each* $x \in X$.

PROOF OF (a). Let $x \in X$, $y \in \mathcal{C}(\alpha; x)$ and let $W \in \Sigma(y)$. Then for any $V \in \Sigma(x)$ we have $W \cap \alpha(V) \neq \varnothing$. Thus $(V \times W) \cap \mathcal{G}(\alpha) \neq \varnothing$. This shows that $(x, y) \in \text{cl}(\mathcal{G}(\alpha))$ and, consequently, that $y \in \pi_y((\{x\} \times Y) \cap \text{cl}(\mathcal{G}(\alpha)))$. The steps in the above proof may be reversed to establish the reverse inclusion.
The proof is complete.

In the first three theorems in this section we characterize multifunctions with closed graphs, strongly-closed graphs and θ-closed graphs in terms of cluster sets. The proofs of Theorems 3.3 and 3.4 are similar to that of Theorem 3.2 and are omitted.

3.2. THEOREM. *The following statements are equivalent for spaces X, Y and $\alpha \in \mathfrak{M}(X, Y)$:*
(a) *The multifunction α has a closed graph*;
(b) $\alpha(x) = \pi_y((\{x\} \times Y) \cap \text{cl}(\mathcal{G}(\alpha)))$ *for each* $x \in X$.
(c) $\mathcal{C}(\alpha; x) = \alpha(x)$ *for each* $x \in X$.

PROOF THAT (a) IMPLIES (b). Obvious.

PROOF THAT (b) IMPLIES (c). This follows from Lemma 3.1 (a).

PROOF THAT (c) IMPLIES (a). Let $(x, y) \in (X \times Y) - \mathcal{G}(\alpha)$. Then $y \notin \alpha(x)$, so $y \notin \mathcal{C}(\alpha; x)$. Thus there are sets $V \in \Sigma(x)$ in X and $W \in \Sigma(y)$ in Y with $\alpha(V) \cap W = \varnothing$. This gives $(V \times W) \cap \mathcal{G}(\alpha) = \varnothing$.

The proof is complete.

3.3. THEOREM. *The following statements are equivalent for spaces X, Y and $\alpha \in \mathfrak{M}(X, Y)$:*

(a) *The multifunction α has a strongly-closed graph*;

(b) $\alpha(x) = \pi_y((\{x\} \times Y) \cap (2)\mathrm{cl}_\theta(\mathcal{G}(\alpha)))$ *for each $x \in X$.*

(c) $\mathcal{S}(\alpha; x) = \alpha(x)$ *for each $x \in X$.*

3.4. THEOREM. *The following statements are equivalent for spaces X, Y and $\alpha \in \mathfrak{M}(X, Y)$:*

(a) *The multifunction α has a θ-closed graph*;

(b) $\alpha(x) = \pi_y((\{x\} \times Y) \cap \mathrm{cl}_\theta(\mathcal{G}(\alpha)))$ *for each $x \in S$.*

(c) $\mathcal{T}(\alpha; x) = \alpha(x)$ *for each $x \in X$.*

Theorem 3.5 gives a characterization of the θ-closure of a subset of a space, and will be utilized in establishing several of our main results.

3.5. THEOREM. *If K is a subset of a space X then $\mathrm{cl}_\theta(K) = \mathrm{ad}\, \Sigma(K)$.*

PROOF. If $V \in \Sigma(K)$ then $\mathrm{cl}_\theta(K) \subset \mathrm{cl}_\theta(V)$; so we have $\mathrm{cl}_\theta(K) \subset \mathrm{cl}(V)$ since the θ-closure of an open set coincides with its closure [**16**]. Thus $\mathrm{cl}_\theta(K) \subset \mathrm{ad}\, \Sigma(K)$. Now suppose $x \notin \mathrm{cl}_\theta(K)$. There is a $W \in \Sigma(x)$ satisfying $x \in W \subset \mathrm{cl}(W) \subset X - K$. It follows that $x \notin \mathrm{ad}\, \Sigma(K)$ since $X - \mathrm{cl}(W) \in \Sigma(K)$. The proof is complete.

Our next results are two of our main results and relate continuity properties of multifunctions to cluster sets.

3.6. THEOREM. *If X and Y are spaces and $\alpha \in \mathfrak{M}(X, Y)$ is u.s.c. then $\mathcal{S}(\alpha; x) = \mathrm{cl}_\theta(\alpha(x))$ for each $x \in X$.*

PROOF. Clearly $\mathrm{cl}_\theta(\alpha(x)) \subset \mathcal{S}(\alpha; x)$ for each $x \in X$. On the other hand, we see that the filterbase $\alpha(\Sigma(x))$ is stronger than the filterbase $\Sigma(\alpha(x))$ in Y since α is u.s.c. Thus $\mathcal{S}(\alpha; x) \subset \mathrm{ad}\, \Sigma(\alpha(x)) = \mathrm{cl}_\theta(\alpha(x))$ (see Theorem 3.5). The proof is complete.

Combining Theorem 3.6 and Theorem 3.3 we easily obtain the following result.

3.7. THEOREM. *An u.s.c. multifunction has a strongly-closed graph if and only if it has θ-closed point images.*

Since θ-closure and closure coincide for subsets of a regular space the following result due to Smithson [**15**] is obtained as a consequence of Theorems 3.7 and 3.2.

3.8. COROLLARY. *If X and Y are spaces with Y regular and $\alpha \in \mathfrak{M}(X, Y)$ is u.s.c. with closed point images, then α has a closed graph.*

Utilizing the fact that a space is T_2 if and only if its points are θ-closed we

reach the following realization of Herrington and Long [9] as a consequence of Theorem 3.7.

3.9. COROLLARY. *A continuous surjection has a strongly-closed graph if and only if it has a Hausdorff image.*

The proof of the following theorem is omitted as it is similar to that of Theorem 3.6.

3.10. THEOREM. *If X and Y are spaces, each w.u.s.c. $\alpha \in \mathfrak{M}(X, Y)$ satisfies $\mathcal{C}(\alpha; x) \subset \mathrm{cl}_\theta(\alpha(x))$ for each $x \in X$.*

Combining Theorem 3.10 with Theorem 3.3 we obtain the following corollary.

3.11. COROLLARY. *A w.u.s.c. multifunction with θ-closed point images has a closed graph.*

Corollary 3.11 generalizes the following result due to Noiri [11].

3.12. COROLLARY. *A weakly-continuous function into a Hausdorff space has a closed graph.*

4. Multifunctions, cluster sets and compactness generalizations. In this section we will characterize compactness, Lindelöfness, m-compactness and H-closedness in terms of multifunctions and cluster sets. We also provide some consequences of these characterizations. We will have need for the next definition.

4.1. DEFINITION. If X and Y are spaces, $K \subset X$ and $\alpha \in \mathfrak{M}(X, Y)$, we define (a) $\mathcal{C}(\alpha; K) = \bigcup_{x \in K} \mathcal{C}(\alpha; x)$, (b) $\mathcal{S}(\alpha; K) = \bigcup_{x \in K} \mathcal{S}(\alpha; x)$ and (c) $\mathcal{T}(\alpha; K) = \bigcup_{x \in K} \mathcal{T}(\alpha; x)$.

The first theorem in this section is preliminary to Theorem 4.3, the first main result in the section. We recall that a subset K of a space X is *m-compact* for an infinite cardinal m if each cover of K by at most m open subsets of X contains a finite subcollection which covers K; it is not difficult to show that K is m-compact if and only if each filterbase Ω on K with cardinality at most m satisfies $K \cap \mathrm{ad}\, \Omega \neq \varnothing$. Theorem 4.2 gives another characterization of m-compact subsets. The proof is omitted.

4.2. THEOREM. *A subset K of a space is m-compact if and only if for each filterbase Ω on the space such that Ω has at most m elements and such that $F \cap V \neq \varnothing$ is satisfied for each $F \in \Omega$ and $V \in \Sigma(K)$, we have $K \cap \mathrm{ad}\, \Omega \neq \varnothing$.*

In the sequel, if X is a nonempty set, $x_0 \in X$ and Ω is a filterbase on X we represent X with the topology $\{A \subset X: x_0 \notin A \text{ or } F \subset A \text{ for some } F \in \Omega\}$ by $X(x_0, \Omega)$. We recall that a space X has *character m* for an infinite cardinal m if for each $x \in X$ there is a local base of open sets at x with cardinality at most m.

4.3. THEOREM. *The following statements are equivalent for a space X and infinite cardinal m:*
   (a) X is m-compact.

(b) $\mathcal{C}(\alpha; K) = \operatorname{ad} \alpha(\Sigma(K))$ for each space Y with character m, $\alpha \in \mathfrak{M}(X, Y)$ and closed $K \subset X$.

(c) $\mathcal{C}(\alpha; K)$ is closed in Y for each space Y with character m, $\alpha \in \mathfrak{M}(X, Y)$ and closed $K \subset X$.

PROOF THAT (a) IMPLIES (b). Let X and Y be any spaces, let $K \subset X$ and let $\alpha \in \mathfrak{M}(X, Y)$. For each $x \in K$ we have $\Sigma(K) \subset \Sigma(x)$, so $\alpha(\Sigma(K)) \subset \alpha(\Sigma(x))$ and, consequently, $\mathcal{C}(\alpha; x) = \operatorname{ad} \alpha(\Sigma(x)) \subset \operatorname{ad} \alpha(\Sigma(K))$ and $\mathcal{C}(\alpha; K) \subset \operatorname{ad} \alpha(\Sigma(K))$. Now let X be m-compact, Y have character m, and let $K \subset X$ be closed. Let $z \in \operatorname{ad} \alpha(\Sigma(K))$ and let Δ be a local base at z of cardinality at most m. Then for each $W \in \Delta$ and $V \in \Sigma(K)$ in X we have $\alpha^{-1}(W) \cap V \neq \varnothing$. So $\alpha^{-1}(\Delta)$ is a filterbase on X which along with the closed (and thus m-compact) subset K satisfies the hypothesis of Theorem 4.2. This implies that $K \cap \operatorname{ad} \alpha^{-1}(\Delta) \neq \varnothing$. For each $x \in K \cap \operatorname{ad} \alpha^{-1}(\Delta)$ we have $V \cap \alpha^{-1}(W) \neq \varnothing$ and, consequently, $\alpha(V) \cap W \neq \varnothing$ for each $V \in \Sigma(x)$ in X and $W \in \Delta$. Thus $z \in \mathcal{C}(\alpha; x)$. The proof that (a) implies (b) is complete.

PROOF THAT (b) IMPLIES (c). Obvious.

PROOF THAT (c) IMPLIES (a). Let Ω be a filterbase of cardinality at most m on X. Let $y_0 \notin X$, let $Y = X \cup \{y_0\}$ and let $\alpha \in \mathfrak{M}(X, Y(y_0, \Omega))$ be the identity function. It is straightforward to check that $Y(y_0, \Omega)$ has character m. Hence from hypothesis $\mathcal{C}(\alpha; X)$ is closed in $Y(y_0, \Omega)$ and we see that $y_0 \in \operatorname{cl}(\mathcal{C}(\alpha; X))$. Thus $y_0 \in \mathcal{C}(\alpha; x)$ for some $x \in X$. For such an x we have $V \cap F = V \cap (F \cup \{y_0\}) \neq \varnothing$ for each $V \in \Sigma(x)$ and $F \in \Omega$. So $\operatorname{ad} \Omega \neq \varnothing$. The proof that (c) implies (a) is complete.

The proof of the theorem is complete.

The following corollary is a consequence of Theorem 4.3.

4.4. COROLLARY. *The following statements are equivalent for a space X:*

(a) *X is compact.*

(b) $\mathcal{C}(\alpha; K) = \operatorname{ad} \alpha(\Sigma(K))$ *for each space Y, $\alpha \in \mathfrak{M}(X, Y)$ and closed $K \subset X$.*

(c) $\mathcal{C}(\alpha; K)$ *is closed in Y for each space Y, $\alpha \in \mathfrak{M}(X, Y)$ and closed $K \subset X$.*

Theorem 4.5 is an analogue of Theorem 4.2 for Lindelöf subsets and is preliminary to Theorem 4.6, another of the main results of this paper.

4.5. THEOREM. *A subset K of a space is Lindelöf if and only if for each filterbase Ω on the space such that $I \cap V \neq \varnothing$ is satisfied by each $V \in \Sigma(K)$ and countable intersection, I, of elements of Ω, we have $K \cap \operatorname{ad} \Omega \neq \varnothing$.*

PROOF. *Sufficiency*. Obvious.

Necessity. Assume that $K \cap \operatorname{ad} \Omega = \varnothing$. Then for each $x \in K$ there is a $V(x) \in \Sigma(x)$ and $F(x) \in \Omega$ such that $V(x) \cap F(x) = \varnothing$; $\{V(x): x \in K\}$ is an open cover of K. Let $K^* \subset K$ be countable such that $\{V(x): x \in K^*\}$ covers K and let $I = \bigcap_{K^*} F(x)$. Then $I \cap \bigcup_{K^*} V(x) = \varnothing$ and $\bigcup_{K^*} V(x) \in \Sigma(K)$. This is a contradiction of the hypothesis on Ω. The proof is complete.

We recall that a space is a $\mathcal{P}$-*space* in the sense of Gillman and Jerison [3] if each G_δ is open.

4.6. THEOREM. *The following statements are equivalent for a space X:*

(a) *X is Lindelöf.*

(b) $\mathcal{C}(\alpha; K) = $ ad $\alpha(\Sigma(K))$ *for each* $\mathcal{P}$*-space* Y, $\alpha \in \mathfrak{M}(X, Y)$ *and closed* $K \subset X$.

(c) $\mathcal{C}(\alpha; K)$ *is closed in* Y *for each* $\mathcal{P}$*-space* Y, $\alpha \in \mathfrak{M}(X, Y)$ *and closed* $K \subset X$.

PROOF THAT (a) IMPLIES (b). We know that $\mathcal{C}(\alpha; K) \subset$ ad $\alpha(\Sigma(K))$ for any spaces X and Y, $K \subset X$ and $\alpha \in \mathfrak{M}(X, Y)$ (see proof of Theorem 4.3). If X is Lindelöf, Y is a $\mathcal{P}$-space, $K \subset X$ is closed and $\alpha \in \mathfrak{M}(X, Y)$, then for each $z \in$ ad $\alpha(\Sigma(K))$ we see that $\alpha^{-1}(\Sigma(x))$ is a filterbase on X which along with the Lindelöf subset K satisfies the hypothesis of Theorem 4.5. This implies that $K \cap$ ad $\alpha^{-1}(\Sigma(z)) \neq \varnothing$. For each $x \in K \cap$ ad $\alpha^{-1}(\Sigma(z))$ we have $V \cap \alpha^{-1}(W) \neq \varnothing$ and, consequently, $\alpha(V) \cap W \neq \varnothing$ for each $V \in \Sigma(x)$ in X and $W \in \Sigma(z)$. Thus $z \in \mathcal{C}(\alpha; x)$. The proof that (a) implies (b) is complete.

PROOF THAT (b) IMPLIES (c). Obvious.

PROOF THAT (c) IMPLIES (a). Let Ω be a filterbase on X with the countable intersection property and let Ω^* be the filterbase of countable intersections of elements of Ω. Choose $y_0 \notin X$, let $Y = X \cup \{y_0\}$ and let $\alpha \in \mathfrak{M}(X, Y(y_0, \Omega^*))$ be the identity function. $Y(y_0, \Omega^*)$ is a $\mathcal{P}$-space. Since ad $\Omega = $ ad Ω^* the rest of the proof proceeds as the proof of (c) implies (a) of Theorem 4.3.

The proof of the theorem is complete.

Corollary 4.7 is a consequence of Theorems 3.2 (c) and 4.3.

4.7. COROLLARY. *Let* X *be a space and* Y *be a space with character* m. *If* $\alpha \in \mathfrak{M}(X, Y)$ *has a closed graph then* $\alpha(K)$ *is closed in* Y *for each* m*-compact* $K \subset X$.

PROOF. $\mathcal{C}(\alpha; K) = \bigcup_{x \in K} \mathcal{C}(\alpha; x) = \bigcup_{x \in K} \alpha(x) = \alpha(K)$. The proof is complete.

4.8. COROLLARY. *Let* X *and* Y *be spaces. If* $\alpha \in \mathfrak{M}(X, Y)$ *has a closed graph then* $\alpha(K)$ *is closed in* Y *for each compact* $K \subset X$.

Corollary 4.9 is a consequence of Theorems 3.2 (c) and 4.6.

4.9. COROLLARY. *If* X *is a space,* Y *a* $\mathcal{P}$*-space and* $\alpha \in \mathfrak{M}(X, Y)$ *has a closed graph, than* $\alpha(K)$ *is closed for each Lindelöf* $K \subset X$.

By the use of techniques similar to those used in the proof of Theorem 4.3, the following theorem may be proved.

4.10. THEOREM. *A space* X *is* m*-compact if and only if* $\mathcal{S}(\alpha; K) = $ ad$_\theta$ $\alpha(\Sigma(K))$ *for each space* Y *with character* m, $\alpha \in \mathfrak{M}(X, Y)$ *and closed* $K \subset X$.

4.11. COROLLARY. *A space* X *is compact if and only if* $\mathcal{S}(\alpha; K) = $ ad$_\theta$ $\alpha(\Sigma(K))$ *for each space* Y, $\alpha \in \mathfrak{M}(X, Y)$ *and closed* $K \subset X$.

The next theorem may also be proved.

4.12. THEOREM. *A space* X *is Lindelöf if and only if* $\mathcal{S}(\alpha; K) = $ ad$_\theta$ $\alpha(\Sigma(K))$ *for each* $\mathcal{P}$*-space* Y, $\alpha \in \mathfrak{M}(X, Y)$ *and closed* $K \subset X$.

4.13. DEFINITION [12]. A subspace A of a space X is *quasi H-closed relative to X* if for each cover θ of A by open subsets of X there is a finite $\theta^* \subset \theta$ with $A \subset \mathrm{cl}(\bigcup_{\theta^*} V)$. If X is quasi H-closed relative to X we say simply that X is *quasi H-closed*. A Hausdorff quasi H-closed space is called *H-closed*.

Herrington [7] has shown that A is quasi H-closed relative to a space X if and only if each filterbase Ω on A satisfies $A \cap \mathrm{ad}_\theta \, \Omega \neq \varnothing$. In Theorem 4.14 we give another characterization of subspaces which are quasi H-closed relative to a space. If X is a space and $K \subset X$, we recall that K is a *regular closed* subset if K is the closure of an open subset of X. We let $\nabla(K)$ denote the collection of regular closed subsets which contain K.

4.14. THEOREM. *A subset K of a space X is quasi H-closed relative to X if and only if for each filterbase Ω on X such that $F \cap C \neq \varnothing$ is satisfied for each $F \in \Omega$ and $C \in \nabla(K)$, we have $K \cap \mathrm{ad}_\theta \, \Omega \neq \varnothing$.*

PROOF. *Sufficiency*. Obvious.

Necessity. Suppose that $K \cap \mathrm{ad}_\theta \, \Omega = \varnothing$. Then for each $x \in K$ there is a $V(x) \in \Sigma(x)$ and an $F(x) \in \Omega$ with $\mathrm{cl}(V(x)) \cap F(x) = \varnothing$. Since K is quasi H-closed relative to X there is a finite $K^* \subset K$ with $K \subset \bigcup_{K^*} \mathrm{cl}(V(x))$. Choose $F^* \in \Omega$ with $F^* \subset \bigcap_{K^*} F(x)$. Then $F^* \cap \bigcup_{K^*} \mathrm{cl}(V(x)) = \varnothing$ and $\bigcup_{K^*} \mathrm{cl}(V(x)) \in \nabla(K)$. Thus Ω does not satisfy the hypothesis.
The proof is complete.

We use Theorem 4.14 together with Theorem 3.5 to prove Theorem 4.15 which improves the result of Veličko [16] that a θ-closed subset of an H-closed space is quasi H-closed relative to the space.

4.15. THEOREM. *If X is a quasi H-closed space and $K \subset X$ then $\mathrm{cl}_\theta(K)$ is quasi H-closed relative to X.*

PROOF. Let Ω be a filterbase on X and suppose that $F \cap C \neq \varnothing$ is satisfied for each $F \in \Omega$ and $C \in \nabla(\mathrm{cl}_\theta(K))$. Then for $V, W \in \bigcup_\Omega \Sigma(F)$ and $C \in \Gamma(K)$ we have $(V \cap W) \cap C \neq \varnothing$. Hence $\Omega^* = \{V \cap Q : V \in \bigcup_\Omega \Sigma(F),\ Q \in \Sigma(K)\}$ is an open filterbase on X; consequently, $\mathrm{ad}\,\Omega^* \neq \varnothing$. Since $\mathrm{ad}\,\Omega^* \subset \mathrm{ad}_\theta \, \Omega \cap \mathrm{ad}\,\Sigma(K)$ and $\mathrm{ad}\,\Sigma(K) = \mathrm{cl}_\theta(K)$ from Theorem 3.5 the proof is complete.

Theorem 4.16 is our next main theorem.

4.16. THEOREM. *The following statements are equivalent for a space X:*

(a) *X is quasi H-closed.*

(b) *$\mathfrak{T}(\alpha; \mathrm{cl}_\theta(K)) \supset \mathrm{ad}_\theta \, \alpha(\nabla(\mathrm{cl}_\theta(K)))$ for each space Y, $\alpha \in \mathfrak{M}(X, Y)$ and $K \subset X$.*

(c) *$\mathfrak{T}(\alpha; K) \supset \mathrm{ad}_\theta \, \alpha(\nabla(K))$ for each space Y, $\alpha \in \mathfrak{M}(X, Y)$ and θ-closed $K \subset X$.*

PROOF THAT (a) IMPLIES (b). If X is quasi H-closed, Y is any space, $K \subset X$ and $\alpha \in \mathfrak{M}(X, Y)$, we see for each $z \in \mathrm{ad}_\theta \, \alpha(\nabla(\mathrm{cl}_\theta(K)))$ that $\alpha^{-1}(\Gamma(z))$ is a filterbase on X which along with $\mathrm{cl}_\theta(K)$ (quasi H-closed relative to X from Theorem 4.15) satisfies the hypothesis of Theorem 4.14. This gives $\mathrm{cl}_\theta(K) \cap \mathrm{ad}_\theta \, \alpha^{-1}(\Gamma(z)) \neq \varnothing$. For each $x \in \mathrm{cl}_\theta(K) \cap \mathrm{ad}_\theta \, \alpha^{-1}(\Gamma(z))$ we have $V \cap \alpha^{-1}(W) \neq \varnothing$ and, consequently, $\alpha(V) \cap W \neq \varnothing$ for each $V \in \Gamma(x)$ in X and $W \in \Gamma(z)$ in Y. Thus $z \in \mathfrak{T}(\alpha; x)$. The proof that (a) implies (b) is complete.

PROOF THAT (b) IMPLIES (c). Obvious.

PROOF THAT (c) IMPLIES (a). Let Ω be a filterbase on X. Let $y_0 \notin X$, let $Y = X \cup \{y_0\}$ and let $\alpha \in \mathfrak{M}(X, Y(y_0, \Omega))$ be the identity function; $\mathrm{ad}_\theta \, \alpha(\nabla(X)) = \mathrm{cl}_\theta(X)$ in $Y(y_0, \Omega)$, X is θ-closed in X and $y_0 \in \mathrm{cl}_\theta(X)$ in $Y(y_0, \Omega)$. Thus by hypothesis we have $y_0 \in \mathfrak{T}(\alpha; X)$. Choose $x \in X$ such that $y_0 \in \mathfrak{T}(\alpha; x)$. Then if $V \in \Gamma(x)$ in X and $F \in \Omega$ we have $F \cup \{y_0\} \in \Gamma(y_0)$ in $Y(y_0, \Omega)$; since $\alpha(V) = V$ we have $V \cap F = V \cap (F \cup \{y_0\}) \neq \varnothing$. The proof that (c) implies (a) is complete.

The proof of the theorem is complete.

Theorem 4.17 is the last one of the main results in this paper.

4.17. THEOREM. *The following statements are equivalent for a Hausdorff space* X:

(a) *X is H-closed.*

(b) *For each space* Y, *$\alpha(\mathrm{cl}_\theta(K))$ is θ-closed in* Y *for each* $K \subset X$ *and* $\alpha \in \mathfrak{M}(X, Y)$ *with a θ-closed graph.*

(c) *For each space* Y, *each* $\alpha \in \mathfrak{M}(X, Y)$ *with a θ-closed graph maps θ-closed subsets onto θ-closed subsets.*

PROOF THAT (a) IMPLIES (b). Let X be H-closed, Y be any space, $\alpha \in \mathfrak{M}(X, Y)$ have a θ-closed graph and $K \subset X$. If $z \in \mathrm{cl}_\theta(\alpha(\mathrm{cl}_\theta(K)))$ then $\alpha^{-1}(\Gamma(z))$ is a filterbase on X which along with $\mathrm{cl}_\theta(K)$, which is quasi H-closed relative to X by Theorem 4.15, satisfies the conditions placed on Ω in Theorem 4.14. Hence $\mathrm{cl}_\theta(K) \cap \mathrm{ad}_\theta \, \alpha^{-1}(\Gamma(z)) \neq \varnothing$. For any $x \in \mathrm{cl}_\theta(K) \cap \mathrm{ad}_\theta \alpha^{-1}(\Gamma(z))$ we have $V \cap \alpha^{-1}(W) \neq \varnothing$ and, consequently, $(V \times W) \cap \mathcal{G}(\alpha) \neq \varnothing$ for each $V \in \Gamma(x)$ in X and $W \in \Gamma(z)$. So $(x, z) \in \mathrm{cl}_\theta(\mathcal{G}(\alpha)) = \mathcal{G}(\alpha)$ and $z \in \alpha(x)$. The proof that (a) implies (b) is complete.

PROOF THAT (b) IMPLIES (c). Obvious.

PROOF THAT (c) IMPLIES (a). Let Ω be a filterbase on X. Employing $Y(y_0, \Omega)$ and $\alpha \in \mathfrak{M}(X, Y(y_0, \Omega))$ as defined in the proof of (c) implies (a) of Theorem 4.16, we see that $y_0 \in \mathrm{cl}_\theta(\alpha(X)) - \alpha(X)$. We conclude that α does not have a θ-closed graph. Since X is Hausdorff there is an $x \in X$ such that $y_0 \in \mathrm{ad}_\theta(\Gamma(x))$ in $Y(y_0, \Omega)$ for $\Gamma(x)$ in X. For such an x, each $V \in \Gamma(x)$ in X and $F \in \Omega$ satisfy $V \cap F \neq \varnothing$, i.e. $x \in \mathrm{ad}_\theta \, \Omega$. The proof that (c) implies (a) is complete.

The proof of the theorem is complete.

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