PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 74, Number 2, May 1979

ALMOST ALL 1-SET CONTRACTIONS HAVE A FIXED POINT

G. J. BUTLER[1]

ABSTRACT. The 1-set contractions and strict set contractions of a bounded, closed, convex subset C of a Banach space X are generalizations of the nonexpansive mappings and the Banach contractions of C, defined in terms of the measure of noncompactness of bounded subsets of X. Vidossich has shown that "almost all" nonexpansive mappings of C into itself have fixed points. In this note we establish a similar generic result for the 1-set contractions of C.

1. Introduction. Let X be a Banach space with norm $\|\cdot\|$. For an arbitrary bounded subset S of X, the measure $\gamma(S)$ of noncompactness of S, introduced by Kuratowski [6] is defined by

$$\gamma(S) = \inf\left\{ \begin{array}{ll} \delta \geqslant 0: & \text{there is a finite cover of } S \\ & \text{by sets of diameter no greater than } \delta \end{array} \right\}.$$

Let C be a fixed, closed, bounded, convex subset of X. If f is a continuous self-map of C and $k \geqslant 0$, f is called a k-set contraction of C if for all $S \subset C$, we have $\gamma(f(S)) \leqslant k\gamma(S)$, where $f(S)$ denotes the image of S under f. Let $(\mathfrak{N}, d)$ be the space of 1-set contractions of C into itself with the metric d defined by $d(f, g) = \sup_{x \in C}\|fx - gx\|$. $\mathfrak{N}$ is a complete metric space containing the set $\mathcal{C}$ of all strict-set contractions of C into itself (k-set contractions with $k < 1$). As particular subspaces of $\mathfrak{N}$, $\mathcal{C}$, respectively, we have the nonexpansive mappings $\mathfrak{M}$ of C and the Banach contractions $\mathcal{K}$ of C, where

$$\mathfrak{M} = \{f: C \to C;\ \|fx - fy\| \leqslant \|x - y\|,\ x, y \in C\},$$

$$\mathcal{K} = \left\{ \begin{array}{ll} f: C \to C; & \text{there exists } k < 1 \text{ such that} \\ & \|fx - fy\| \leqslant k\|x - y\|,\ x, y \in C \end{array} \right\}.$$

For any class $\mathcal{A}$ of self-maps of C, let $\mathcal{F}(\mathcal{A})$ denote the subclass of those maps f of $\mathcal{A}$ which have a fixed point in C, i.e., $fx_0 = x_0$ for some $x_0 \in C$. Of course we have $\mathcal{F}(\mathcal{K}) = \mathcal{K}$ and it is well known that $\mathcal{F}(\mathcal{C}) = \mathcal{C}$ [3], and if X is uniformly convex, $\mathcal{F}(\mathfrak{M}) = \mathfrak{M}$ [1], [4], [5].

Vidossich [8] has shown that for any arbitrary Banach space X, almost all (in the sense of category) nonexpansive self-mappings of C have fixed points; that is $\mathcal{F}(\mathfrak{M})$ is residual in $\mathfrak{M}$ (contains a countable intersection of open

Received by the editors August 25, 1977.
AMS (MOS) subject classifications (1970). Primary 54H25, 54E40.
Key words and phrases. Fixed points, k-set contractions, genericity.
[1]Supported in part by NRC grant no. A-8130.

dense subsets of $\mathfrak{M}$). Recently, De Blasi and Myjak [2] have proved the even stronger result that there is a residual subset $\mathfrak{M}_0$ of $\mathfrak{M}$ such that for all $f \in \mathfrak{M}_0$, f has a unique fixed point x_0 and $f^n x \to x_0$ as $n \to \infty$, for all $x \in C$.

In this note we shall extend the Vidossich result to the 1-set contractions of C into itself, that is we shall prove

THEOREM. *$\mathfrak{F}(\mathfrak{N})$ is residual in $\mathfrak{N}$.*

2. Definitions and preliminary results. For subsets S of C, we shall denote their closure by $\bar{S}$ and their convex closure by $\mathrm{co}(\bar{S})$, and for $r \geqslant 0$, $B(S; r)$ will be $\{x \in C\colon \|x - y\| \leqslant r \text{ for some } y \in S\}$. Thus if $S = \{x_0\}$, $B(x_0; r)$ is the closed ball about x_0 of radius r, and if $r = 0$, $B(S; r) = S$. We recall the definition of the Hausdorff metric on the set of nonempty subsets of C: if $S, T \subset C$, $S \neq \varnothing \neq T$, then

$$\rho(S, T) = \inf\{\varepsilon \geqslant 0\colon S \subset B(T; \varepsilon) \text{ and } T \subset B(S; \varepsilon)\}.$$

The following lemma lists some well-known results involving the measure γ of noncompactness (see [7]).

LEMMA 1. *Let S be a bounded, nonempty subset of X. Then*

(a) $\gamma(S) = 0$ *iff* $\bar{S}$ *is compact.*

(b) $\gamma(B(S; r)) \leqslant \gamma(S) + 2r$.

(c) $\gamma(\mathrm{co}(\bar{S})) = \gamma(S)$.

(d) *Let $A_0 \supset A_1 \supset A_2 \supset \dots$ be a decreasing sequence of closed, nonempty subsets of C, such that* $\lim_{n\to\infty} \gamma(A_n) = 0$. *Then $A = \bigcap_{n=0}^{\infty} A_n$ is compact and nonempty, and* $\lim_{n\to\infty} A_n = A$ *in the Hausdorff metric.*

Let $g \in \mathcal{C}$ (the strict-set contractions of C) and let $\varepsilon > 0$. We define the sequences $D_n^\varepsilon = D_n^\varepsilon(g)$, $n = 0, 1, \dots,$ as follows:

$$D_0^\varepsilon = C,\ D_n^\varepsilon = \overline{\mathrm{co}\left(B\left(g\left(D_{n-1}^\varepsilon\right); \varepsilon\right)\right)}, \qquad n = 1, 2, \dots.$$

We make the following observations:

(i) For each $\varepsilon \geqslant 0$, $D_0^\varepsilon \supset D_1^\varepsilon \supset D_2^\varepsilon \supset \dots$ is a decreasing sequence of closed, nonempty subsets of C.

(ii) If $0 \leqslant \varepsilon \leqslant \eta$, then $D_n^\varepsilon \subset D_n^\eta$, $n = 0, 1, 2, \dots$.

For $\varepsilon = 0$, we have the following

LEMMA 2.

(a) *$K = \bigcap_{m=0}^{\infty} D_m^0$ is a nonempty, compact, convex subset of C, invariant under g, and g has a fixed point in K.*

(b) *Given a natural number n, there exists a natural number $m_1 = m_1(g, n)$ such that $D_m^0 \subset B(K; 1/n)$, for $m \geqslant m_1$.*

PROOF. (a) was first shown by Darbo [3] and is a consequence of Lemma 1 and the Schauder fixed point theorem. (b) follows from (a), and part (d) of Lemma 1.

Next we establish a kind of continuity result for D_m^ε as $\varepsilon \to 0$ and $m \to \infty$.

LEMMA 3. *Let $g \in \mathcal{C}$, $K = K(g)$ as defined in Lemma 2, and let n be a natural number. Then there exists a natural number $m_2 = m_2(g, n)$ such that for all $p \geqslant m \geqslant m_2$ we have $D_p^{1/m} \subset B(K; 1/n)$. Thus $\lim_{\varepsilon \to 0, m \to \infty} D_m^{\varepsilon} = K$ in the Hausdorff metric.*

PROOF. g is a k-set contraction of C for some $k < 1$. Let $\gamma(C) = c$. Then $\gamma(g(C)) \leqslant k\gamma(C) = kc$, and so $\gamma(\overline{\text{co}(B(g(C); 1/m))}) \leqslant kc + 2/m$, by Lemma 1, that is $\gamma(D_1^{1/m}) \leqslant kc + 2/m$. It is easily established by induction on p that (1) $\gamma(D_p^{1/m}) \leqslant k^p c + (2/m)(1/(1-k))$, $p = 1, 2, \ldots$. Let $D = D(g) = \bigcap_{m=1}^{\infty} D_m^{1/m}$. We note that $D_m^{1/m}$ is a decreasing sequence of closed, convex, nonempty subsets of C, and by (1),

$$\gamma\left(D_m^{1/m}\right) = k^m c + (2/m)(1/(1-k)) \to 0 \quad \text{as } m \to \infty.$$

By Lemma 1(d), D is a nonempty, compact, convex set and $\lim_{m \to \infty} D_m^{1/m} = D$ in the Hausdorff metric. D also contains K since $D_m^{1/m} \supset D_m^0 \supset K$, $m = 1, 2, \ldots$. Thus it will suffice to show that $D = K$ and then apply Lemma 2(b) in order to obtain Lemma 3.

Since $D_m^{1/m}$ is invariant under g for each m, it follows that D is invariant under g. Since D is also compact and convex, we have $D \supset \overline{\text{co}(g(D))}$. Let $\varepsilon > 0$ be given. Since g is continuous and D is compact, there exists $\eta = \eta(\varepsilon) > 0$ such that for any $x, y \in B(D; \eta)$ with $\|x - y\| \leqslant \eta$, we have $\|gx - gy\| \leqslant \varepsilon/2$. Then $g(B(D; \eta)) \subset B(g(D); \varepsilon/2)$, and since $D_m^{1/m} \subset B(D; \eta)$ for m sufficiently large, we have $g(D_m^{1/m}) \subset B(g(D); \varepsilon/2)$ and so

$$\begin{aligned} D_{m+1}^{1/m+1} \subset D_{m+1}^{1/m} &= \overline{\text{co}\left(B\left(g\left(D_m^{1/m}\right); 1/m\right)\right)} \\ &\subset \overline{\text{co}\left(B\left(g(D); \varepsilon/2 + 1/m\right)\right)} \subset \text{co}\left(\overline{B\left(g(D); \varepsilon\right)}\right) \end{aligned}$$

for m sufficiently large.Thus $D = \bigcap_{m=1}^{\infty} D_m^{1/m} \subset \overline{\text{co}(B(g(D); \varepsilon))}$. Since this holds for arbitrary $\varepsilon > 0$ and since $\bigcap_{\varepsilon > 0} \overline{\text{co}(B(g(D); \varepsilon))} = \overline{\text{co}(g(D))}$, we find that $D \subset \overline{\text{co}(g(D))} \subset D$, and so $D = \overline{\text{co}(g(D))}$. We have $D \subset C$ and therefore $D = \overline{\text{co}(g(D))} \subset \overline{\text{co}(g(C))} = D_1^0$ and inductively we find $D \subset D_m^0$, $m = 0, 1, 2, \ldots$. Thus we have $D \subset \bigcap_{m=0}^{\infty} D_m^0 = K \subset D$, and so $D = K$. The lemma now follows.

LEMMA 4. *Let $g \in \mathcal{C}$. Let $K = K(g)$, $m_2 = m_2(g; n)$, $n = 1, 2, \ldots$, be as in Lemma 3. Then if $h \in \mathcal{C}$ with $d(h, g) < 1/m_2$, we have $K(h) \subset B(K(g); 1/n)$.*

PROOF. For all $x \in C$, we have $\|hx - gx\| < 1/m_2$. Therefore $h(C) \subset B(g(C); 1/m_2)$, and so

$$D_1^0(h) = \text{co}\left(\overline{h(C)}\right) \subset \text{co}\left(\overline{B(g(C); 1/m_2)}\right) = D_1^{1/m_2}(g).$$

Similarly, we have $h(D_1^0(h)) \subset B(D_1^{1/m_2}(g); 1/m_2)$, and we inductively obtain $D_m^0(h) \subset D_m^{1/m_2}(g)$, $m = 1, 2, \ldots$. Thus for $m \geqslant m_2$, we have

$$D_m^0(h) \subset D_m^{1/m_2}(g) \subset B(K(g); 1/n)$$

and so $K(h) \subset B(K(g); 1/n)$.

Before proceeding to the proof of the theorem, we note the trivial

LEMMA 5. *$\mathcal{C}$ is dense in $\mathfrak{N}$.*

PROOF. W.l.o.g. $O \in C$. Then for any $f \in \mathfrak{N}$, $\varepsilon > 0$, $g = (1 - \varepsilon/2)f$ is a $(1 - \varepsilon/2)$-set contraction of C and $d(f, g) < \varepsilon$.

3. Proof of the theorem. We modify the arguments used in **[2]**, **[8]**. For each $g \in \mathcal{C}$ and each natural number n, let $K(g)$, $m_2 = m_2(g, n)$ be as in Lemma 4. Define $U_n(g)$ to be $\{f \in \mathfrak{N}: d(f, g) < 2^{-n}m_2^{-1}\}$ and define $\mathfrak{N}_0$ to be $\bigcap_{n=1}^{\infty} \bigcup_{g \in \mathcal{C}} U_n(g)$. By Lemma 5, $\mathfrak{N}_0$ is a dense G_δ subset of $\mathfrak{N}$ and to prove the theorem it remains to show that $\mathfrak{N}_0 \subset \mathfrak{F}(\mathfrak{N})$.

Let $f \in \mathfrak{N}_0$, and let $g_n \in \mathcal{C}$ be such that $f \in U_n(g_n)$, $n = 1, 2, \ldots$. Define n_1 to be 1 and choose $n_2 \geqslant 2$ so large that $2^{-n_2}(m_2(g_{n_2}, n_2))^{-1} < 2^{-1}(m_2(g_1, 1))^{-1}$. Then $d(g_1, g_{n_2}) \leqslant d(g_1, f) + d(f, g_{n_2}) < (m_2(g_1, 1))^{-1}$. It follows from Lemma 4 that $K_2 \subset B(K_1; 1)$ where $K_1 = K(g_1)$, $K_2 = K(g_{n_2})$. Inductively, if $n_1, \ldots, n_j$ have been chosen, choose $n_{j+1} \geqslant 2n_j$ sufficiently large that

$$2^{-n_{j+1}}\left(m_2(g_{n_{j+1}}, n_{j+1})\right)^{-1} < 2^{-n_j}\left(m_2(g_{n_j}, n_j)\right)^{-1}.$$

We find that

$$d(g_{n_j}, g_{n_{j+1}}) < 2^{1-n_j}\left(m_2(g_{n_j}, n_j)\right)^{-1} < \left(m_2(g_{n_j}, n_j)\right)^{-1}$$

and so $K_{j+1} \subset B(K_j; 1/n_j)$, where $K_i = K(g_{n_i})$, $i = 1, 2, \ldots, j + 1$. Let $r_j = \sum_{i=j}^{\infty} 1/n_i$. Then $r_j \leqslant \sum_{i=j}^{\infty} 2^{1-i} < \infty$, $j = 1, 2, \ldots$. We note that $B(K_{j+1}; r_{j+1}) \subset B(K_j; r_j)$, $j = 1, 2, \ldots$. Choose $x_j \in K_j$ such that $g_{n_j}x_j = x_j$, and let $S_j = \overline{\{x_i\}_{i=j}^{\infty}}$. Then $S_1 \supset S_2 \supset \ldots$ is a decreasing sequence of nonempty closed subsets of C, and $S_j \subset B(K_j; r_j)$, $j = 1, 2, \ldots$. Therefore $\gamma(S_j) \leqslant 2r_j \to 0$ as $j \to \infty$, and hence from Lemma 1(d), there exists a point $x_0 \in \bigcap_{j=1}^{\infty} S_j$, and so there is a subsequence which we again label x_j such that $\lim_{j\to\infty} x_j = x_0$. Thus $\lim_{j\to\infty} fx_j = fx_0$. On the other hand,

$$\begin{aligned} \|fx_j - x_j\| &\leqslant \|g_{n_j}x_j - x_j\| + \|fx_j - g_{n_j}x_j\| \\ &< 2^{-n_j}\left(m_2(g_{n_j}, n_j)\right)^{-1} \to 0 \text{ as } \quad j \to \infty. \end{aligned}$$

It follows that $fx_0 = x_0$, and so $f \in \mathfrak{F}(\mathfrak{N})$. The proof of the theorem is complete.

REFERENCES

1. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041–1044.

2. F. S. De Blasi and J. Myjak, *Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach*, C. R. Acad. Sci. Paris Sér. A **283** (1976), 185–187.

3. G. Darbo, *Punti uniti in transformazione a condiminio non compatto*, Rend. Sem. Mat. Univ. Padova **24** (1955), 84–92.

4. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr. **30** (1966), 251–258.

5. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006.

6. C. Kuratowski, *Sur les espaces complets*, Fund. Math. **15** (1930), 301–309.

7. R. D. Nussbaum, *The fixed point index for local condensing maps*, Ann. Mat. Pura Appl. **89** (1971), 217–258.

8. G. Vidossich, *Existence, uniqueness and approximation of fixed points as a generic property*, Bol. Soc. Brasil. Mat. **5** (1974), 17–29.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1