

ROTUND COMPLEX NORMED LINEAR SPACES

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ABSTRACT. We show that rotundity in a complex normed linear space is equivalent to the property that for any distinct vectors x and y of unit norm, a complex number α may be found for which $\|\alpha x + (1 - \alpha)y\| < 1$. This leads to a natural proof of a result due to Taylor and Foguel on the uniqueness of Hahn-Banach extensions.

A normed linear space $(X, \|\cdot\|)$ is *rotund* (or *strictly convex*) if whenever $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ we have $\|\frac{1}{2}(x + y)\| < 1$. It is well known that in this case we actually have $\|tx + (1 - t)y\| < 1$ for $0 < t < 1$. Moreover (cf. A. E. Taylor [1, p. 544]) it is easy to see that if X is *not* rotund there exists a real line segment $tx + (1 - t)y$, $0 \leq t \leq 1$ ($x \neq y$), all of whose points lie on the surface of the unit ball, i.e., $\|tx + (1 - t)y\| = 1$, $0 \leq t \leq 1$.

All of this applies whether X is a real or complex normed linear space. In the case that X is a complex linear space we shall prove the following alternative characterization of rotund spaces. Geometrically the characterization states that if X is *not* rotund there exists a complex line $\alpha x + (1 - \alpha)y$, $\alpha \in \mathbb{C}$ ($x \neq y$, $\|x\| = \|y\| = 1$), all of whose points lie outside, or on the surface of, the unit ball.

THEOREM 1. *A normed linear space $(X, \|\cdot\|)$ over the complex field \mathbb{C} is rotund if, and only if, X has the property (L): whenever $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ we have $\|\alpha x + (1 - \alpha)y\| < 1$ for some $\alpha \in \mathbb{C}$.*

PROOF. Clearly if X is rotund then the property (L) holds with $\alpha = \frac{1}{2}$, so it is enough to prove the converse.

Suppose, then, that X is a complex normed linear space. We shall prove that if X is not rotund then X does not have property (L). Suppose that $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = \|(x + y)/2\| = 1$. By the Hahn-Banach theorem, there is an $x^* \in X^*$ such that $\|x^*\| = 1 = x^*((x + y)/2)$. So $x^*x/2 + x^*y/2 = 1$ and $|x^*x| \leq 1$, $|x^*y| \leq 1$, and since 1 is an extreme point of the unit disk it follows that $x^*x = x^*y = 1$. Hence for any complex number α ,

$$\|\alpha x + (1 - \alpha)y\| \geq |x^*(\alpha x + (1 - \alpha)y)| = 1,$$

and so property (L) fails.

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We remark that the above proof uses the Hahn-Banach theorem, but the axiom of choice is not really required, because a simple argument shows that it is sufficient to prove Theorem 1 for spaces of two (complex) dimensions, and so only the finite-dimensional Hahn-Banach theorem is needed. Even in two dimensions it appears difficult to give a purely geometric proof.

As an application of Theorem 1 we prove the following result due to Taylor [1, Theorem 6] and Foguel [2].

THEOREM 2. *Let X be a complex normed linear space. Then all bounded linear functionals defined on subspaces of X have unique norm-preserving linear extensions to X if, and only if, the conjugate space X^* is rotund.*

PROOF. Suppose X^* is rotund but that not all Hahn-Banach extensions are unique. That is, there is a subspace V of X and a bounded linear functional v^* on V having distinct linear extensions $x^*, y^* \in X^*$ with $\|x^*\| = \|y^*\| = \|v^*\|$. We may assume that $\|v^*\| = 1$, clearly. Now $z^* = \alpha x^* + (1 - \alpha)y^*$ is a linear extension of v^* for all $\alpha \in C$. By rotundity we have

$$\|\alpha x^* + (1 - \alpha)y^*\| < 1$$

for some $\alpha \in C$. This is impossible since, for all α ,

$$\|\alpha x^* + (1 - \alpha)y^*\| \geq \sup_{\substack{v \in V \\ \|v\|=1}} |v^*(v)| = 1.$$

Conversely, suppose that X has the unique Hahn-Banach extension property. To see that X^* is rotund, take any $x^*, y^* \in X^*$ with $x^* \neq y^*$, $\|x^*\| = \|y^*\| = 1$. Let

$$V = \ker(x^* - y^*) = \{x \in X: x^*x = y^*x\},$$

and let v^* be the restriction of x^* (or y^*) to V ; v^* has a unique extension $z^* \in X^*$ with $\|z^*\| = \|v^*\| < 1$. This last inequality follows from the unique-extension property and the fact that x^* and y^* both extend v^* . Fix any $x_0 \in X - V$, so $x^*x_0 \neq y^*x_0$. Hence the system of equations

$$\alpha x^*x_0 + \beta y^*x_0 = z^*x_0, \quad \alpha + \beta = 1,$$

has a unique solution $(\alpha, \beta) \in C^2$, so there exists $\alpha \in C$ such that

$$\alpha x^*x_0 + (1 - \alpha)y^*x_0 = z^*x_0.$$

Since $X = \text{Sp}\{x_0, V\}$ it follows that $\alpha x^* + (1 - \alpha)y^* = z^*$, and we now have

$$\|\alpha x^* + (1 - \alpha)y^*\| = \|z^*\| < 1,$$

so that X^* is rotund by Theorem 1.

REFERENCES

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