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HYPERINVARIANT SUBSPACES OF C_{11} CONTRACTIONS

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ABSTRACT. For an operator T on a Hilbert space let Hyperlat T denote its hyperinvariant subspace lattice. Assume that T is a completely nonunitary C_{11} contraction with finite defect indices. In this note we characterize the elements of Hyperlat T among invariant subspaces for T in terms of their corresponding regular factorizations and show that elements in Hyperlat T are exactly the spectral subspaces of T defined by Sz.-Nagy and Foiaş. As a corollary, if T_1, T_2 are two such operators which are quasi-similar to each other, then Hyperlat T_1 is (lattice) isomorphic to Hyperlat T_2.

1. Introduction. Let T be a bounded linear operator acting on a complex separable Hilbert space H. A subspace K of H is *hyperinvariant* for T if K is invariant for every operator that commutes with T. We denote by Hyperlat T the lattice of all hyperinvariant subspaces of T. Recently several authors studied Hyperlat T for certain classes of contractions. Uchiyama showed that Hyperlat T is preserved, as a lattice, for quasi-similar $C_0(N)$ contractions and for completely injection-similar $C_{\cdot 0}$ contractions with finite defect indices (cf. [6] and [7]). As a result he was able to determine Hyperlat T indirectly for such contractions. Wu, in [8], determined Hyperlat T when T is a completely nonunitary (c.n.u.) contraction with a scalar-valued characteristic function or a direct sum of such contractions. In this note we investigate Hyperlat T for c.n.u. C_{11} contractions with finite defect indices. Our main result (Theorem 1) says that for such contractions elements in Hyperlat T are exactly the spectral subspaces H_F defined by Sz.-Nagy and Foiaş in [5]. Thus we can completely determine Hyperlat T in terms of the well-known structure of the hyperinvariant subspace lattice of normal operators. As a corollary, we show that for such contractions Hyperlat T is preserved, as a lattice, under quasi-similarities.

2. Preliminaries. A contraction T is *completely nonunitary* (*c.n.u.*) if there exists no nontrivial reducing subspace on which T is unitary. The *defect indices* of T are, by definition,

$$d_T = \operatorname{rank}(I - T^*T)^{\frac{1}{2}} \quad \text{and} \quad d_{T^*} = \operatorname{rank}(I - TT^*)^{\frac{1}{2}}.$$

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$T \in C_{\cdot 1}$ (resp. $C_{1\cdot}$) if $T^{*n}x \not\to 0$ (resp. $T^n x \not\to 0$) for all $x \neq 0$; $C_{11} = C_{\cdot 1} \cap C_{1\cdot}$. For a C_{11} contraction T, $d_T = d_{T^*}$. Let Θ_T denote the characteristic function of an arbitrary contraction T. There is a one-to-one correspondence between the invariant subspaces of T and the regular factorizations of Θ_T. If $K \subseteq H$ is invariant for T with the corresponding regular factorization $\Theta_T = \Theta_2\Theta_1$ and $T = \left[\begin{smallmatrix} T_1 & X \\ 0 & T_2 \end{smallmatrix}\right]$ is the triangulation on $H = K \oplus K^{\perp}$, then the characteristic functions of T_1, T_2 are the purely contractive parts of Θ_1, Θ_2, respectively. For more details the readers are referred to **[5]**.

For arbitrary operators T_1, T_2 on H_1, H_2, respectively, $T_1 \prec T_2$ denotes that there exists a one-to-one operator X from H_1 onto a dense linear manifold of H_2 (called *quasi-affinity*) such that $XT_1 = T_2X$. T_1, T_2 are *quasi-similar* ($T_1 \sim T_2$) if $T_1 \prec T_2$ and $T_2 \prec T_1$. For any subset E of the unit circle C, let M_E denote the operator of multiplication by e^{it} on the space $L^2(E)$ of square-integrable functions on E. It was proved in **[9]** that any c.n.u. C_{11} contraction T with finite defect indices is quasi-similar to a uniquely determined operator, called the *Jordan model* of T, of the form $M_{E_1} \oplus \cdots \oplus M_{E_k}$, where $E_1, \ldots, E_k$ are Borel subsets of C satisfying $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_k$. In this case $E_1 = \{t: \Theta_T(t) \text{ not isometric}\}$.

We use t to denote the argument of a function defined on C. A statement involving t is said to be true if it holds for almost all t with respect to the Lebesgue measure. In particular, for $E, F \subseteq C$, $E = F$ means that $(E \setminus F) \cup (F \setminus E)$ has Lebesgue measure zero. For any subset F of C, $F' \equiv C \setminus F$.

3. Main results. We start with the following

LEMMA 1. *Let T be a C_{11} contraction on H and U be a unitary operator on K. If there exists a one-to-one operator X: $H \to K$ such that $XT = UX$, then T is quasi-similar to the unitary operator $U|_{\overline{XH}}$.*

PROOF. Since T, being a C_{11} contraction, is quasi-similar to a unitary operator, the assertion follows from Lemma 4.1 of **[2]** immediately.

Let T be a c.n.u. C_{11} contraction on H with finite defect indices and let $U = M_{E_1} \oplus \cdots \oplus M_{E_k}$ acting on $K = L^2(E_1) \oplus \cdots \oplus L^2(E_k)$ be its Jordan model. Let X: $H \to K$ and Y: $K \to H$ be quasi-affinities intertwining T and U. For any Borel subset $F \subseteq E_1$, let

$$K_F = L^2(E_1 \cap F) \oplus \cdots \oplus L^2(E_k \cap F)$$

be the spectral subspace of K associated with F. For the contraction T we considered, $\sigma(T) \subseteq C$ holds and there has been developed a spectral decomposition (cf. **[5**, p. 318 and pp. 315–316, resp.**]**). Let H_F denote the spectral subspace associated with $F \subseteq C$. Indeed, H_F is the (unique) maximal subspace of H satisfying (i) $TH_F \subseteq H_F$, (ii) $T_F \equiv T|_{H_F} \in C_{11}$ and (iii) $\Theta_{T_F}(t)$ is isometric for t in F'. Moreover H_F is hyperinvariant for T. We shall show that such subspaces H_F give all the elements in Hyperlat T. We prove this in a series of lemmas.

LEMMA 2. *For any Borel subset* $F \subseteq E_1$, $\overline{XH_F} = K_F$.

PROOF. Let $K_1 = \overline{XH_F}$. Since $T_F \equiv T|_{H_F}$ is of class C_{11}, Lemma 1 implies that T_F is quasi-similar to the unitary operator $U|_{K_1}$. Consider K as a subspace of L^2_k in the natural way. Hence K_1 is a reducing subspace for the bilateral shift M on L^2_k. From the well-known structure of reducing subspaces of M, we obtain that $K_1 = PL^2_k$, where P is a measurable function from C to the set of (orthogonal) projections on $\mathbf{C}^k$. Since

$$K_1 \subseteq K = L^2(E_1) \oplus \cdots \oplus L^2(E_k),$$

we have

$$P(t)\mathbf{C}^k \subseteq \mathbf{C}^j \oplus \underbrace{0 \oplus \cdots \oplus 0}_{k-j}$$

for t in $E_j \setminus E_{j+1}, j = 1, \ldots, k-1$, and $P(t) = 0$ for t in E_1'. For almost all t, let $\{\psi_j(t)\}_1^k$ be an orthonormal base of $\mathbf{C}^k$ consisting of eigenvectors of $P(t)$, that is, such that

$$P(t)\psi_j(t) = \delta_j(t)\psi_j(t), \quad j = 1, \ldots, k,$$

where the eigenvalues $\delta_j(t)$ are arranged in nonincreasing order: $1 \geqslant \delta_1(t) \geqslant \cdots \geqslant \delta_k(t) \geqslant 0$ (cf. [5, p. 272]). Let

$$F_j = \{t\colon \text{rank } P(t) \geqslant j\} = \{t\colon \delta_j(t) > 0\} \quad \text{for } j = 1, \ldots, k.$$

Then $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k$, $E_j \supseteq F_j$ and $P(t)\psi_j(t) = \chi_{F_j}(t)\psi_j(t)$ for each j. Setting $x_j(t) = (v(t), \psi_j(t))$ for $v \in L^2_k$ where $(\,,\,)$ denotes the usual inner product in $\mathbf{C}^k$, we have $v(t) = \sum_1^k x_j(t)\psi_j(t)$. Since for $v \in K_1$,

$$v(t) = P(t)v(t) = \sum_1^k \chi_{F_j}(t)x_j(t)\psi_j(t),$$

the induced transformation

$$v \to x_1\chi_{F_1} \oplus \cdots \oplus x_k\chi_{F_k}$$

maps K_1 isometrically onto $L^2(F_1) \oplus \cdots \oplus L^2(F_k)$ (cf. [5, p. 272]). Moreover $U|_{K_1}$ will be carried over by this transformation to $M_{F_1} \oplus \cdots \oplus M_{F_k}$. We infer that $F_1 = \{t\colon \Theta_{T_F}(t) \text{ not isometric}\} \subseteq F$ (cf. the remark in §2). Thus for $v \in K_1$, $v(t) = \sum_1^k \chi_{F_j}(t)x_j(t)\psi_j(t) = 0$ on F', which shows that $v \in K_F$, and hence $K_1 \subseteq K_F$.

To show the other inclusion, let $x \in K_F$ and $K_2 = \overline{XH_{F'}}$. Since $H = H_F \vee H_{F'}$, we have $K = K_1 \vee K_2$. Hence there exist sequences $\{y_n\} \subseteq K_1$ and $\{z_n\} \subseteq K_2$ such that $y_n + z_n \to x$. From what we proved above, $\{y_n\} \subseteq K_F$ and $\{z_n\} \subseteq K_{F'}$. Since $K = K_F \oplus K_{F'}$, by applying the (orthogonal) projection onto K_F on both sides of $y_n + z_n \to x$ we obtain $y_n \to x$. This shows that $x \in K_1$, completing the proof.

For any Borel subset $F \subseteq E_1$, let $q(K_F) = \bigvee_{ST=TS} SYK_F$. It is known that $q(K_F)$ is hyperinvariant for T and $\overline{Xq(K_F)} = K_F$ (cf. [5, pp. 76–78]).

LEMMA 3. *For any Borel subset $F \subseteq E_1$, let $q(K_F)$ be defined as above. Then $q(K_F) = H_F$.*

PROOF. Let $\Theta_T = \Theta_2\Theta_1$ be the regular factorization corresponding to $q(K_F)$. To complete the proof it suffices to show that (i) Θ_1 is outer, (ii) $\Theta_1(t)$ is isometric for t in F' and (iii) $\Theta_2(t)$ is isometric for t in F (cf. [5, pp. 312 and 205]). Since $q(K_F) \in$ Hyperlat T, $\sigma(T|_{q(K_F)}) \subseteq \sigma(T)$ (cf. [1, Lemma 3.1]). It follows that $T|_{q(K_F)}$ is also of class C_{11} (cf. [5, p. 318]), and hence Θ_1 is outer (from both sides). This proves (i).

Since $\overline{Xq(K_F)} = K_F$ and $YK_F \subseteq q(K_F)$, on the decompositions $H = q(K_F) \oplus q(K_F)^{\perp}$ and $K = K_F \oplus K_{F'}$, X, Y, T and U can be triangulated as

$$X = \begin{bmatrix} X_1 & * \\ 0 & X_2 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 & * \\ 0 & Y_2 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}, \quad U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}.$$

It is easily seen that X_1 is a quasi-affinity intertwining T_1, U_1, so that $T_1 \prec U_1$. Since $T_1 = T|_{q(K_F)}$ is a C_{11} contraction, we conclude from Lemma 1 that $T_1 \sim U_1$. This shows that $U_1 = \sum_{j=1}^{k} \oplus M_{E_j \cap F}$ is the Jordan model of T_1, and hence $F = E_1 \cap F = \{t\colon \Theta_1(t) \text{ not isometric}\}$. Therefore $F' = \{t\colon \Theta_1(t) \text{ isometric}\}$, which proves (ii). On the other hand, X_2^* and Y_2^* are one-to-one operators intertwining T_2^*, U_2^*. Note that T_2 is also of class C_{11}. (This follows from the fact that $\det \Theta_2 \not\equiv 0$ and [5, p. 318].) Let V be the unitary operator quasi-similar to T_2. We infer that there are one-to-one operators intertwining V^*, U_2^*. It follows from Lemma 4.1 of [2] that V^* and U_2^* are unitarily equivalent to direct summands of each other. By the third test problem in [4] we conclude that V^*, U_2^*, and hence V, U_2, are unitarily equivalent. So $T_2 \sim U_2$. A similar argument as above shows that $E_1 \cap F' = \{t\colon \Theta_2(t) \text{ not isometric}\}$. Hence $E_1' \cup F = \{t\colon \Theta_2(t) \text{ isometric}\}$, which proves (iii) and completes the proof.

LEMMA 4. *Let $\mathfrak{M} \subseteq H$ be hyperinvariant for T with the corresponding factorization $\Theta_T = \Theta_2\Theta_1$ and let $F = \{t\colon \Theta_1(t) \text{ not isometric}\}$. Then $\mathfrak{M} = H_F$.*

PROOF. As proved in Lemma 3, for hyperinvariant $\mathfrak{M}$, $T|_{\mathfrak{M}}$ is of class C_{11}. Since $\Theta_{T|_{\mathfrak{M}}}(t)$ is isometric for t in F', the maximality of H_F implies that $\mathfrak{M} \subseteq H_F$; cf. the remark before Lemma 2. Hence $\overline{X\mathfrak{M}} \subseteq \overline{XH_F} = K_F$, by Lemma 2. We claim that $K_F = \bigvee_{SU=US} S\overline{X\mathfrak{M}}$. Indeed, using Lemma 1 we can show that $T|_{\mathfrak{M}}$ is quasi-similar to $U|_{\overline{X\mathfrak{M}}}$. Now we proceed as in the proof of Lemma 2 with $\overline{X\mathfrak{M}}$ in the role of K_1. Let P be a projection-valued function defined on C such that $\overline{x\mathfrak{M}} = PL_k^2$. Choose the orthonormal base $\{\psi_j(t)\}_1^k$ of $\mathbf{C}^k$ consisting of eigenvectors of $P(t)$, and let $F_j = \{t\colon \operatorname{rank} P(t) \geq j\}$ for $j = 1, \ldots, k$. Note that for $v \in L_k^2$, $v = \sum_1^k x_j\psi_j$, where $x_j(t) = (v(t), \psi_j(t))$ for each j and $v = \sum_1^k \chi_{F_j} x_j \psi_j$ if $v \in \overline{X\mathfrak{M}}$. As shown before, the transformation $v \to \chi_{F_1}x_1 \oplus \cdots \oplus \chi_{F_k}x_k$ maps $\overline{X\mathfrak{M}}$ isometrically onto $L^2(F_1) \oplus \cdots \oplus L^2(F_k)$, and hence $M_{F_1} \oplus \cdots \oplus M_{F_k}$ is the Jordan model of $T|_{\mathfrak{M}}$. We have $F_1 = \{t\colon \Theta_{T|_{\mathfrak{M}}}(t) \text{ not isometric}\} = F = E_1 \cap F$. For each j, let S_j be the operator on K defined by

$$S_j(v) = 0 \oplus \cdots \oplus \chi_{E_j \cap F}\, x_1 \oplus \cdots \oplus 0$$

for $v = \sum_1^k x_j \psi_j \in K$. It is easily seen that $S_j U = U S_j$ and

$$\overline{S_j \overline{X \mathfrak{M}}} = 0 \oplus \cdots \oplus L^2(E_j \cap F) \oplus \cdots \oplus 0$$

for each j. It follows that $K_F = \bigvee_{SU=US} \overline{SX\mathfrak{M}}$, as asserted. By Lemma 3,

$$H_F = q(K_F) = \bigvee_{VT=TV} VYK_F = \bigvee_{VT=TV} \bigvee_{SU=US} VYS\overline{X\mathfrak{M}}.$$

Since $VYSX$ commutes with T and $\mathfrak{M}$ is hyperinvariant for T, we have $H_F \subseteq \mathfrak{M}$. This, together with $\mathfrak{M} \subseteq H_F$, completes the proof.

Now we have the following main theorem.

THEOREM 1. *Let T be a c.n.u. C_{11} contraction on H with $d_T = d_{T^*} = n < \infty$. Let $K \subseteq H$ be an invariant subspace with the corresponding regular factorization $\Theta_T = \Theta_2\Theta_1$ and let $E = \{t\colon\ \Theta_T(t)$ not isometric$\}$. Then the following are equivalent*:

(1) $K \in$ Hyperlat T;

(2) *$K = H_F$ for some Borel subset $F \subseteq E$;*

(3) *the intermediate space of $\Theta_T = \Theta_2\Theta_1$ is of dimension n and for almost all t, either $\Theta_2(t)$ or $\Theta_1(t)$ is isometric.*

PROOF. (1) $\Rightarrow$ (2). That $K = H_F$, where $F = \{t\colon\ \Theta_1(t)$ not isometric$\}$, is proved in Lemma 4. It is a simple matter to check that $F \subseteq E$.

(2) $\Rightarrow$ (3). Since $T|_{H_F} \in C_{11}$, the intermediate space of $\Theta_T = \Theta_2\Theta_1$ is of dimension n (cf. [5, p. 192]). The rest is proved in [5, p. 312].

(3) $\Rightarrow$ (1). Since the intermediate space of $\Theta_T = \Theta_2\Theta_1$ is of dimension n and $\det \Theta_1 \not\equiv 0$ (otherwise $\det \Theta_T \equiv 0$), we conclude that $T|_K$ is of class C_{11} (cf. [5, p. 318]). Therefore, Θ_1 is outer (from both sides). This, together with the other condition in (3), implies that $K = H_F$, where $F = \{t\colon\ \Theta_1(t)$ not isometric$\}$ (cf. [5, p. 312]). Thus $K \in$ Hyperlat T.

COROLLARY 1. *Let T be as in Theorem 1 and let $U = M_{E_1} \oplus \cdots \oplus M_{E_k}$, acting on K, be its Jordan model. Then* Hyperlat T *is* (*lattice*) *isomorphic to* Hyperlat U. *Moreover, if $X\colon H \to K$ and $Y\colon K \to H$ are quasi-affinities intertwining T, U, then the mapping $\mathfrak{M} \to \overline{X\mathfrak{M}}$ implements the lattice isomorphism from* Hyperlat T *onto* Hyperlat U, *and its inverse is given by $\mathfrak{N} \to q(\mathfrak{N}) = \bigvee_{ST=TS} SY\mathfrak{N}$. In this case, $T|_{\mathfrak{M}}$ and $U|_{\overline{X\mathfrak{M}}}$ are quasi-similar to each other.*

PROOF. The first assertion follows from Theorem 1, [5, pp. 315–316] and the well-known structure of Hyperlat U [3]. The rest are immediate consequences of Lemmas 1, 2 and 3.

COROLLARY 2. *Let T_1, T_2 be c.n.u. C_{11} contractions with finite defect indices. If T_1 is quasi-similar to T_2, then* Hyperlat T_1 *is* (*lattice*) *isomorphic to* Hyperlat T_2.

COROLLARY 3. *Let T be a c.n.u. C_{11} contraction with finite defect indices. If $K_1, K_2 \in$ Hyperlat T and $T|_{K_1}$ is quasi-similar to $T|_{K_2}$, then $K_1 = K_2$.*

PROOF. $T|_{K_1} \sim T|_{K_2}$ implies that they have the same Jordan model, say, $U = M_{E_1} \oplus \cdots \oplus M_{E_k}$. By Theorem 1, $K_1 = H_{E_1} = K_2$.

ADDED IN PROOF. After submitting this paper, the author was notified that the main result here was independently obtained by R. I. Teodorescu (*Factorisations régulières et sous-espaces hyperinvariants*, to appear in Acta Sci. Math. (Szeged)) for arbitrary c.n.u. C_{11} contractions. However the approaches are completely different.

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