PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 1, June 1979

CRITERIA FOR FUNCTIONS TO BE OF HARDY CLASS H^p

SHINJI YAMASHITA

ABSTRACT. Let f be holomorphic in the disk $|z| < 1$. Two criteria (see (I) and (II)) for f to be of H^2 are extended to the case of H^p, $0 < p < +\infty$, by the methods different from known ones for $p = 2$.

1. Results. By f we always mean a function holomorphic in $D = \{|z| < 1\}$. We set for $0 < p < +\infty$,

$$f_p^* = \frac{p}{2}\,|f|^{p/2-1}|f'|;$$

roughly speaking, f_p^* is the absolute value of the derivative of $f^{p/2}$ which has no meaning when f has a zero in D. Obviously, $f_2^* = |f'|$. We shall show that f_p^*, which may assume the value $+\infty$, plays important roles for f to be of class H^p $(0 < p < +\infty)$ in D. Here, f is said to be of Hardy class H^p $(0 < p < +\infty)$ if

$$I_p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta$$

is bounded for $0 \leqslant r < 1$ (see [**1**, p. 2]). Set

$$A_p(f) = \iint_D (1 - |z|) f_p^*(z)^2\, dx\, dy \qquad (z = x + iy).$$

The following (I) is observed in [**5**, Remark (a), p. 208].

(I) *f is of class H^2 if and only if $A_2(f) < +\infty$.*

This is the case $p = 2$ in our

THEOREM 1. *Let f be a function holomorphic in D, and let $0 < p < +\infty$. Then f is of class H^p if and only if $A_p(f) < +\infty$.*

Our proof of Theorem 1 is different from that of (I) (see [**5**, p. 208]) where the coefficients of the Taylor expansion of f about 0 play the essential roles.

Now, let G be a subdomain of D such that the boundary of G has the only one point 1 in common with the unit circle. Assume that there exists r_0, $0 < r_0 < 1$, depending on G, such that the intersection of G with each circle $\{|z| = r\}$, $r_0 < r < 1$, is of linear measure $r\varphi(r)$, where

$$\liminf_{r\to 1}\,(1 - r)^{-1}\varphi(r) > 0 \tag{1.1}$$

Received by the editors May 23, 1978.
AMS (MOS) subject classifications (1970). Primary 30A78.

0002-9939/79/0000-0264/$02.00

and

$$\limsup_{r \to 1} (1 - r)^{-1} \varphi(r) < +\infty. \tag{1.2}$$

Let $\mathcal{G}$ be the family of all domains G of the type described above. A typical example of G is a triangular domain in D with one vertex at 1, which we shall call, for short, a triangular domain at 1. Of course, $G \in \mathcal{G}$ may look like a "Swiss cheese", having many holes. Denoting

$$G(\theta) = \{ z \in D \mid e^{-i\theta} z \in G \}, \qquad \theta \in [0, 2\pi],$$

we say that f satisfies the p-Lusin property with respect to $G \in \mathcal{G}$ if

$$L_p(f, G, \theta) = \iint_{G(\theta)} f_p^*(z)^2 \, dx \, dy$$

is summable with respect to θ on $[0, 2\pi]$ $(0 < p < +\infty)$; thus, it follows that $L_p(f, G, \theta) < +\infty$ for almost every θ in $[0, 2\pi]$.

G. Piranian and W. Rudin [4, Theorem 1] reformulated (in effect, with an addition) N. Lusin's theorem [3] as follows.

(II) *If $f \in H^2$, then f has the 2-Lusin property with respect to each triangular domain at 1. Conversely, if f has the 2-Lusin property with respect to a certain triangular domain at 1, then $f \in H^2$.*

Again, this is the case $p = 2$ in our

THEOREM 2. *Let f be a function holomorphic in D, and let $0 < p < +\infty$. If $f \in H^p$, then f has the p-Lusin property with respect to each domain of class $\mathcal{G}$. Conversely, if f has the p-Lusin property with respect to a certain domain of $\mathcal{G}$, then $f \in H^p$.*

Our proof of Theorem 2 is different from that of (II) (see [3] and [4]) where the coefficients of the Taylor expansion of f about 0 play the essential roles.

2. Proof of Theorem 1. First of all, the local consideration of f_p^*, near the zeros of f, shows that for each $0 < p < +\infty$ and for each $0 < r < 1$,

$$E_p(r, f) = \iint_{|z| < r} f_p^*(z)^2 \, dx \, dy < +\infty.$$

In effect, G. H. Hardy and P. Stein (see [2, Theorem 3.1, p. 42]) proved much more:

$$r \frac{d}{dr} I_p(r, f) = \frac{2}{\pi} E_p(r, f), \qquad 0 < r < 1.$$

It then follows from the integration that

$$I_p(r, f) - |f(0)|^p = \frac{2}{\pi} \int_0^r E_p(t, f) t^{-1} \, dt, \qquad 0 < r < 1. \tag{2.1}$$

Especially,

$$\int_0^r E_p(t, f) t^{-1} \, dt < +\infty \tag{2.2}$$

for each $0 < r < 1$. Now, it follows from (2.1) that, $f \in H^p$ if and only if

$$\int_0^1 E_p(t, f)t^{-1}\, dt < +\infty. \tag{2.3}$$

In view of (2.2) one can conclude from (2.3) that, $f \in H^p$ if and only if

$$\int_0^1 E_p(r, f)\, dr < +\infty. \tag{2.4}$$

Now, letting $X_r(z)$ be the characteristic function of the disk $\{|z| < r\}$, $0 < r < 1$, one observes that

$$\int_0^1 X_r(z)\, dr = 1 - |z|, \qquad z \in D,$$

whence

$$\begin{aligned}\int_0^1 E_p(r, f)\, dr &= \int_0^1 dr \iint_D f_p^*(z)^2 X_r(z)\, dx\, dy \\ &= \iint_D \left[\int_0^1 X_r(z)\, dr\right] f_p^*(z)^2\, dx\, dy = A_p(f).\end{aligned}$$

This equality, together with (2.4), completes the proof of Theorem 1.

3. Proof of Theorem 2. Let $G \in \mathcal{G}$ with r_0 and φ as described in the definition of G. Letting $X(z, \theta)$ be the characteristic function of $G(\theta)$, $\theta \in [0, 2\pi]$, one observes that

$$\int_0^{2\pi} X(z, \theta)\, d\theta, \qquad z \in D,$$

is the linear measure of the set $\{e^{i\theta} | z \in G(\theta)\}$. Therefore,

$$\varphi(|z|) = \int_0^{2\pi} X(z, \theta)\, d\theta, \qquad r_0 < |z| < 1. \tag{3.1}$$

On the other hand,

$$\begin{aligned}L_p(f, G) &\equiv \int_0^{2\pi} L_p(f, G, \theta)\, d\theta \\ &= \iint_D \left[\int_0^{2\pi} X(z, \theta)\, d\theta\right] f_p^*(z)^2\, dx\, dy.\end{aligned} \tag{3.2}$$

It follows from (1.1), (1.2), (3.1), and (3.2) that $L_p(f, G) < +\infty$ if and only if there exists r_2, $r_0 < r_2 < 1$, such that

$$\iint_{r_2<|z|<1} (1 - |z|) f_p^*(z)^2\, dx\, dy < +\infty. \tag{3.3}$$

Since (3.3) is equivalent to $A_p(f) < +\infty$, Theorem 2 now follows from Theorem 1.

REFERENCES

1. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York and London, 1970.

2. W. K. Hayman, *Multivalent functions*, Cambridge Univ. Press, London and New York, 1958.

3. N. Lusin, *Sur une propriété des fonctions à carré sommable*, Bull. Calcutta Math. Soc. **20** (1930), 139–154.

4. G. Piranian and W. Rudin, *Lusin's theorem on areas of conformal maps*, Michigan Math. J. 3 (1955–1956), 191–199.

5. A. Zygmund, *Trigonometric series*. II, Cambridge Univ. Press, London and New York, 1959.

DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, FUKAZAWA, SETAGAYA-KU, TOKYO 158, JAPAN