OPEN NONNEGATIVELY CURVED 3-MANIFOLDS WITH A
POINT OF POSITIVE CURVATURE

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ABSTRACT. Let M be a complete open nonnegatively curved Riemannian
3-manifold with a point at which all sectional curvatures are positive, and
suppose that M contains a pole. Then M is not flat on the complement of
any compact set. Note that this is clearly false for 2-manifolds.

In this short note I will prove the following: Let M^3 be a complete open
nonnegatively curved 3-manifold with a point at which all sectional
curvatures are positive. Suppose further that M contains a pole. Then M is
not flat off any compact set.

Despite the ease with which this is proven, I feel that this formal
presentation is warranted on several grounds. Among others, these include
the following: (1) the result is, perhaps, counterintuitive–it is clearly false in
two dimensions; (2) it opens up similar questions in higher dimensions which
cannot be so easily answered; (3) finally, if the need for a pole represents a
defect in the proof and not a real restriction, which I believe to be the case, it
would lead to one of the few instances in which a geometric condition at one
point implies a global geometric (not topological) result.

Notation and preliminary remarks. (1) M will always denote a complete
Riemannian manifold.

(2) A point p in M is called a *pole* (see [2]) if the exponential map $\exp_p$:
$M_p \to M$ is a submersion or has maximal rank on all M_p. If p is a pole, $\exp_p$
will be a diffeomorphism if M is simply connected.

(3) If M^3 is an open nonnegatively curved 3-manifold with a point of
positive curvature, then (see [1]) M is diffeomorphic to $\mathbf{R}^3$; in particular it is
simply connected.

(4) $T_r(A)$ will denote the open tubular neighborhood of radius r about the
set $A \subset M$, and $S_r(A)$ will denote $\partial T_r(A)$, the sphere of radius r about A.
N.B. $S_r(A)$ is contained in M, not TM.

(5) If $x \in M$ and γ is a normal geodesic ray in M with $\gamma(0) = x$, then
$H_x(\gamma)$ will denote the complementary half-space determined by x and γ; i.e.
$H_x(\gamma) = M \setminus \{ y \in M | d(\gamma(t), y) < t, \ t \in [0, \infty) \}$. Recall that $H_x(\gamma)$ is
totally convex if M is nonnegatively curved (see [1]).

(6) If C is an oriented codimension one submanifold of M with orientation
vector field N, i.e., N is a unit length section of the normal bundle of C in

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TM, let II_N be the second fundamental form on C relative to N, so that $\mathrm{II}_N(X, Y) = \langle \nabla_X N, Y \rangle$ for X and Y in TC.

(7) We call a q-form with values in p-forms a (q, p)-form. For a résumé of the calculus of (q, p)-forms, see for example [3].

It would seem that the following two lemmas, although surely known in some quarters, do not appear in the literature. Hence I have included brief proofs.

LEMMA 1. *Let M be a complete open nonnegatively curved manifold, and let $p \in M$ be a pole. If $x \in S_r(p)$ and γ is the ray originating at p and passing through x, let $N(x) = \gamma'(x)$ determine an orienting vector field N for $S_r(p)$. Then II_N is positive semidefinite on $S_r(p)$.*

PROOF. Let x and γ be as in the statement of the lemma, and suppose that $\gamma(0) = p$, $\gamma(a) = x$. Then $\gamma|_{[a,\infty)}$ is a ray originating at x, and so the complementary half space $H_x(\gamma)$ may be constructed. Clearly $T_r(p) \subset H_x(\gamma)$, and so the support plane for $H_x(\gamma)$ at x is a support plane for $T_r(p)$ at x. Since $H_x(\gamma)$ is at least locally convex, the lemma follows. $\square$

REMARK. Using the same underlying idea one could easily give an elementary, if somewhat longer, proof of this lemma using only the Rauch comparison theorem.

LEMMA 2. *Let M^n be a complete open nonnegatively curved manifold. Let $C \subset M$ be an oriented codimension one submanifold with orientation vector field N. Extend N to a neighborhood of C by unit speed geodesics; i.e., if $y = \exp_x t_0 N(x)$ for $x \in C$, let $N(y) = (\exp_x t N(x))'(t_0)$. Furthermore suppose that II_N is positive semidefinite on C. Set $C_t = \{y | y = \exp_x t N(x)$ for some $x \in C\}$. Then $(d/dt)(\int_{C_t} \det \mathrm{II}_N)_{t=\rho} \leq 0$ as long as C_t is smooth and II_N is ≥ 0 on C_t for all $0 \leq t \leq \rho$.*

PROOF. Regard II_N as a $(1, 1)$-form with $\mathrm{II}_N(X)(Y) = \langle \nabla_X N, \cdot \rangle(Y) = \langle \nabla_X N, Y \rangle$. Then we can write $\int_{C_t} \det \mathrm{II}_N = \int_{C_t} \langle N, \cdot \rangle \wedge \mathrm{II}_N^{n-1}$, where $\langle N, \cdot \rangle$ is a $(0, 1)$-form, and thus $\langle N, \cdot \rangle \wedge \mathrm{II}_N^{n-1}$ is an $((n-1), n)$-form, and thus is integrable over C_t. Furthermore,

$$\int_{C_t} \langle N, \cdot \rangle \wedge \mathrm{II}_N^{n-1} - \int_{C_{t-\epsilon}} \langle N, \cdot \rangle \wedge \mathrm{II}_N^{n-1} = \int_{D_{t,\epsilon}} d(\langle N, \cdot \rangle \wedge \mathrm{II}_N^{n-1}),$$

where $D_{t,\epsilon} = T_t(C) \setminus T_{t-\epsilon}(C)$. But an easy computation yields

$$\int_{D_{t,\epsilon}} d(\langle N, \cdot \rangle \wedge \mathrm{II}_N^{n-1}) = \int_{D_{t,\epsilon}} \mathrm{II}_N^n + (n-1)\int_{D_{t,\epsilon}} \langle N, \cdot \rangle \wedge \Omega_N \wedge \mathrm{II}_N^{n-2},$$

where Ω_N is the $(2, 1)$-form $\Omega_N(X, Y)(Z) = \langle R(X, Y)N, \cdot \rangle(Z) = \langle R(X, Y)N, Z \rangle$.

Since $\nabla_N N = 0$, $\mathrm{II}_N^n = 0$ on $D_{t,\epsilon}$. Furthermore, if $X_1, \ldots, X_n$ is a local orthonormal basis of eigenvectors of II_N with eigenvalues $\lambda_1, \ldots, \lambda_n$, and $X_1 = N$, and if $K_{ij} = -\langle R(X_i, X_j)X_i, X_j \rangle$, then

$$\langle N, \cdot \rangle \wedge \Omega_N \wedge \text{II}_N^{n-2}(X_1, \ldots, X_n)(X_1, \ldots, X_n)$$

$$= (n - 2)! \sum_{j=2}^{n} - \left[K_{1j} \left(\prod_{\substack{i \neq 1 \\ i \neq j}} \lambda_i \right) \right].$$

But since $K_{1j} \geqslant 0$ and $\lambda_i \geqslant 0$ for all $0 \leqslant i, j \leqslant n$,

$$\int_{C_t} \langle N, \cdot \rangle \wedge \text{II}_N^{n-1} - \int_{C_{t-\epsilon}} \langle N, \cdot \rangle \wedge \text{II}_N^{n-1} < 0. \quad \square$$

PROPOSITION. *Let M^3 be a complete open nonnegatively curved 3-manifold with a point q in M at which all sectional curvatures are positive. Suppose further that M contains a pole p. Then M is not flat off any compact set.*

PROOF. Let r be any real number and let X and Y denote vector fields over $S_r(p)$. Let $K_r(X, Y)$ denote the sectional curvature in M of the section spanned by X and Y, and let $\overline{K}_r(X, Y)$ denote the curvature in the induced metric in $S_r(p)$ of the section spanned by X and Y. Then it is standard that $K_r(X, Y) = \overline{K}_r(X, Y) - \det \text{II}(X, Y)$. Furthermore, the Gauss-Bonnet theorem states that $\int_{S_r(p)} \overline{K}_r = 4\pi$, so that

$$\int_{S_r(P)} K_r = 4\pi - \int_{S_r(p)} \det \text{II}.$$

But Lemma 2 then implies that $(d/dt)\int_{S_r(p)} K_r \geqslant 0$. Hence if $r \geqslant d(p, q)$, $\int_{S_r(p)} K_r > 0$. $\quad \square$

REMARK. Note that, since $\exp_p$ is volume decreasing ($K > 0$), one obtains a bound on the rate at which sectional curvature density can decrease as r increases.

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