

ON RECURRENT RANDOM WALKS ON
 ABELIAN SEMIGROUPS

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ABSTRACT. We consider a random walk Z_n on the locally compact second countable abelian semigroup S with unit e which is assumed to be a recurrent point for Z_n . Then S is the disjoint union of a topologically simple abelian semigroup G and a null-set A of first category. Under some additional conditions G is a topological group.

Let S be a multiplicatively written locally compact, noncompact, second countable abelian semigroup with unit e and let μ be a regular probability measure the support of which generates S . Let $\{Z_n | n = 0, 1, 2, \dots\}$ be the random walk on S generated by μ . Our basic assumption is that e is recurrent, i.e.

$$\Pr\{Z_n \in U \text{ infinitely often} | Z_0 = e\} = 1 \tag{1}$$

for every neighborhood U of e .

We will show that (1) implies that S is the disjoint union of G and A where G is a topologically simple semigroup, i.e., G has no proper closed ideals, and A is a set of first category.

Put $\nu = \sum_1^\infty 2^{-n} \mu^n$ where μ^n is the n -fold convolution product of μ with itself. The support of ν is all of S .

Write $x \rightarrow y$ if y can be reached from x , i.e. $\nu(x^{-1}N) > 0$ for every neighborhood N of y ; here $x^{-1}N \equiv \{s \in S | xs \in N\}$.

LEMMA. *Let A be the set $\{y \in S | y \nrightarrow e\}$. Then A is a first category ν -null-set.*

PROOF. Let $\{U_n\}$ be a neighborhood basis at e . Then A is the union of closed sets $A_n \equiv \{y \in S | \nu(y^{-1}U_n) = 0\}$. A_n is closed since $\nu(y^{-1}U_n)$ is a lower semicontinuous function of y . $y \in A_n$ is equivalent to $yS \cap U_n = \emptyset$ which shows that A_n is an absorbing set. As in [6, p. 142] one sees that $\mu^k(A_n) > 0$ for some k implies that $\sum_k \mu^k(A_n^c)$ converges. In particular, $\sum_k \mu^k(U_n)$ converges which contradicts (1). ((1) together with the joint continuity of the multiplication implies that $x \rightarrow x$ for all x .) Thus $\mu^k(A_n) = 0$ for all k whence $\nu(A_n) = 0$ and $\nu(A) = 0$. Since the support of ν is all of S no ν -null-set can have interior points. Hence A is of first category.

Suppose first that A is not dense in S . Then A is closed and it is a maximal closed ideal in S . To see this, let U be an open set in A^c . Then $\bar{A} \subset U^c$. Let

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$\{a_n\}$ be a sequence in A converging to $a \in S$. Then $\overline{\{a_n\}S} \subset \overline{A} \subset U^c$ whence $\overline{aS} \subset U^c$. Every element of S is reached from e , so if $x \rightarrow e$ then $x \rightarrow s$ for all $s \in S$, which is equivalent to $\overline{xS} = S$. Hence e cannot be reached from a , i.e. $a \in A$ and A is closed. A is clearly an ideal. Let B be a proper closed ideal of S and take $b \in B$. Then bS ($\subset B \neq S$) is not dense in S , so $b \in A$ and $B \subset A$.

If A is compact then the Rees quotient, see [5], S/A is a topologically O -simple topological semigroup. In general, however, S/A is only an algebraic semigroup (the multiplication in S/A is not necessarily jointly continuous).

Let $G = S \setminus A$. G is a semigroup: For any $x \in G$, $x^{-1}A$ is a closed ideal of S , so it is either a subset of A or equal to S . But $e \notin x^{-1}A$. Hence $x^{-1}A \subset A$ which means that $xy \in G$ if x and y belong to G . G is also a stochastically closed set since $\mu(x^{-1}A) = 0$ for all $x \in G$.

Restrict Z_n to the stochastically closed set G which is locally compact and second countable in the induced topology. (1) implies that $\sum_n \mu^n(N) = \infty$ for every neighborhood of e . If $ye = y \in N$ then $yN' \subset N$ for some suitable neighborhood N' of e . Hence $\sum_n \mu^n(y^{-1}N) = \infty$ for all $y \in N$. For $x \in G$ there is a k with $\mu^k(x^{-1}N) \equiv P^k(x, N) > 0$. Clearly

$$P^{n+k}(x, N) \geq \int_N P^k(x, dy) P^n(y, N).$$

Consequently, we have

$$\sum_n P^n(x, N) = \infty \quad \text{for all } x \in G \text{ and all neighborhoods } N \text{ of } e. \quad (2)$$

In particular, N can be chosen to be relatively compact. By [8, p. 174] there is then an invariant σ -finite measure π , finite on compact subsets of G . xG is cancellative since xG is dense in G for all $x \in G$, cf. [4, p. 18]. If π is a probability measure then it is idempotent and G is a compact group, see [3]. Hence A is empty in this case. In the general case we get the following result.

PROPOSITION. *Let A be closed. Then G is a topologically simple semigroup admitting a recurrent random walk. The following statements are equivalent:*

- (i) G is a topological group.
- (ii) For all $x \in G$, xG is closed (in G).
- (iii) For all $x \in G$, there is a neighborhood N of e such that $x^{-1}N$ is relatively compact.
- (iv) For all $x \in G$, xG is of second category in G .
- (v) For all $x \in G$, xG has nonempty interior.
- (vi) For all $x \in G$, $\pi(G \setminus xG) = 0$.

PROOF. If (ii) or (iii) are valid then x has an inverse whence (i) by [2, p. 38]. Hence the first three statements are equivalent.

G is σ -compact so xG is a countable union of compact sets. By Baire's category theorem, (iv) and (v) are equivalent. (ii) is equivalent to

(ii)' xG contains an idempotent.

The equivalence of (ii)' and (vi) is shown in [1, Theorem 2.7]. It remains, then, to show that (v) implies (ii)' (since clearly (i) implies (v)). Let U be an open subset of xG . Then $xx^{-1}U = U$ and, for $u \in U$, Uu^{-1} is a neighborhood of e such that

$$(xx^{-1}U)u^{-1} = x(x^{-1}Uu^{-1}) = Uu^{-1},$$

since $x^{-1}Uu^{-1}$ is nonempty. Hence $e \in xG$.

REMARK 1. If A is dense in S then G is a countable intersection of open sets. G may be equipped with a complete metric [7]. Hence G is a topological group if any of the conditions (ii)–(vi) are satisfied. ((i)–(vi) are not equivalent in this case; for instance, (i) does not imply (iii).)

REMARK 2. Finally we remark that if additional conditions of continuity (for instance: there is an x_0 such that for any Borel set B the function $P(x, B)$ is continuous at x_0) are imposed then A will be closed and the conditions in the Proposition satisfied, see [1].

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