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SHORTER NOTES

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AN INTERSECTION RESULT FOR TENSOR PRODUCTS OF C^*-ALGEBRAS

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ABSTRACT. There exists a C^*-tensor product $A \otimes B$ with C^*-subalgebras $A_1 \otimes B_1$ and $A_2 \otimes B_2$ such that $(A_1 \otimes B_1) \cap (A_2 \otimes B_2)$ strictly contains $(A_1 \cap A_2) \otimes (B_1 \cap B_2)$.

Let A and B be C^*-algebras and let $A \otimes B$ denote their minimal (spatial) C^*-tensor product. For each $\varphi \in A^*$ there exists a unique bounded linear map R_φ: $A \otimes B \to B$ satisfying $R_\varphi(a \otimes b) = \langle \varphi, a \rangle b$. Similarly, for each $\psi \in B^*$ there exists a unique bounded linear map L_ψ: $A \otimes B \to A$ satisfying $L_\psi(a \otimes b) = \langle \psi, b \rangle a$.

Let C and D be C^*-subalgebras of A and B, respectively, and put

$$F(C, D) = \{x \in A \otimes B: R_\phi(x) \in D, L_\psi(x) \in C \ (\phi \in A^*, \psi \in B^*)\}.$$

We consider A and B canonically embedded in their enveloping W^*-algebras A^{**} and B^{**}, so that $A \otimes B$ is contained in $A^{**} \otimes B^{**}$ canonically. Let C^- and D^- be the weak closures of C and D.

THEOREM. *With the above situation, if $F(C, D)$ strictly contains $C \otimes D$ and there exist projections of norm one from A^{**} and B^{**} onto C^- and D^-, respectively, then $(A \otimes B) \cap (C^- \otimes D^-)$ strictly contains $(A \cap C^-) \otimes (B \cap D^-)$.*

PROOF. By [2, Proposition 3.7], we have

$$C^- \otimes D^- = \{x \in A^{**} \otimes B^{**}: R_\phi(x) \in D^-, \quad L_\psi(x) \in C^- \ (\phi \in A^{***}, \psi \in B^{***})\}.$$

Let $x \in (A \otimes B) \cap (C^- \otimes D^-)$. If $\varphi \in A^{***}$, then we get

$$R_\phi(x) = R_{(\phi|A)}(x) \in B \cap D^- = D.$$

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Similarly, if $\psi \in B^{***}$, we get

$$L_\psi(x) = L_{(\psi|B)}(x) \in C.$$

It follows that

$$(A \otimes B) \cap (C^- \otimes D^-) \subseteq F(C, D).$$

The opposite inclusion is clearly true. Thus we obtain

$$(A \otimes B) \cap (C^- \otimes D^-) = F(C, D) \supsetneq C \otimes D = (A \cap C^-) \otimes (B \cap D^-).$$

If D is a closed two-sided ideal of B, then D^- is a two-sided ideal of B^{**}. Thus there exists a unique central projection z in B^{**} such that $D^- = B^{**}z$ (see, for example, [1, 1.10.5]). The map $x \to xz$ is a projection of norm one from B^{**} onto D^-.

If $A = C$, a C^*-subalgebra D is a two-sided ideal of B, and $F(C, D)$ strictly contains $C \otimes D$, then the quadruplet (A, B, C, D) of C^*-algebras satisfies the conditions of the theorem by the preceding remark. Wassermann ([4], [5]) gave several such quadruplets of C^*-algebras. Hence we obtain examples requested by Wassermann [3, Remark 23].

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