MARTINGALE REPRESENTATION IN
UNIFORMLY CONVEX SPACES

D. LANDERS AND L. ROGGE

ABSTRACT. In this paper we define the concept of a martingale in a uniformly convex Banach space and show that each bounded martingale is convergent and can be represented as a sequence of nearest point projections onto closed convex sets of one element of the Banach space. This result contains classical representation theorems for usual martingales and yields applications in the theory of nearest point projections in L_p-spaces.

1. Notations. Let B be a Banach space with norm $\| \ \|$. If $x \in B$, $C \subset B$, let $d(x, C) = \inf\{\|x - y\|: y \in C\}$.

The Banach space B is uniformly convex iff for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\|x_1\| \leqslant 1$, $\|x_2\| \leqslant 1$, $\|\frac{1}{2}(x_1 + x_2)\| \geqslant 1 - \delta(\varepsilon)$ implies $\|x_1 - x_2\| \leqslant \varepsilon$.

For the sake of completeness we cite the following result, which can easily be derived from the considerations in [1, pp. 113–114].

REMARK. Let B be a uniformly convex Banach space and $r > 0$ be fixed. Then for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all convex sets $C \subset B$ with $0 \in C$ we have: $\|x\| \leqslant r, y_1, y_2 \in C$ and $\|x - y_1\|, \|x - y_2\| \leqslant d(x, C) + \delta(\varepsilon)$ imply $\|y_1 - y_2\| \leqslant \varepsilon$.

In a uniformly convex space B for each $x \in B$ and each closed convex set $C \subset B$ there exists a unique projection $\pi(x|C) \in C$ of x onto C, i.e. a unique element $\pi(x|C) \in C$ such that $\|x - \pi(x|C)\| = d(x, C)$.

DEFINITION. We call $x_n \in B$, $n \in \mathbf{N}$, a martingale with respect to the closed convex sets $C_n \subset B$, $n \in \mathbf{N}$, iff

(i) $0 \in C_m \subset C_n$ for $m \leqslant n$,

(ii) $x_m = \pi(x_n|C_m)$ for $m \leqslant n$.

If B is a Hilbert space this is the definition of Brunk [3].

If B is an $L_p(\Omega, \mathcal{C}, P)$, where $(\Omega, \mathcal{C}, P)$ is a probability space, then important examples for projections onto closed convex sets are the classical conditional expectations, the conditional expectations given a σ-lattice (see [2], [3]), the p-predictors in the sense of Ando-Amemiya (see [1]) or the conditional p-means in the sense of Brunk (see [4]).

2. The result. Now we give a representation theorem for martingales, which generalizes Brunk's representation theorem (see [3]) from Hilbert spaces to

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uniformly convex Banach spaces. We remark that Brunk's methods are closely adjusted to a Hilbert space and cannot be applied for uniformly convex Banach spaces.

THEOREM. *Let B be a uniformly convex Banach space. Let $x_n \in B$, $n \in \mathbf{N}$, be a martingale with respect to the closed convex sets $C_n \subset B$, $n \in \mathbf{N}$. Assume that $\sup\|x_n\| < \infty$. Then x_n, $n \in \mathbf{N}$, converges strongly to an element $x \in B$ and $x_n = \pi(x|C_n)$, $n \in \mathbf{N}$.*

PROOF. Since a uniformly convex Banach space is reflexive and in a reflexive Banach space each bounded set is relatively weakly sequentially compact there exists a subsequence x_n, $n \in \mathbf{N}_0$, converging weakly to an element $x \in B$.

Since $\sup_{n \in \mathbf{N}}\|x_n\| < \infty$, we may w.l.g. assume that $\lim_{n \in \mathbf{N}_0}\|x_n - x\| = a$, $a \in \mathbf{R}$.

We prove at first that

$$x_n, n \in \mathbf{N}_0, \text{ is Cauchy convergent.} \tag{$*$}$$

As for convex sets weak closure and strong closure coincide and x is in the weak closure of the convex set $\bigcup_{n \in \mathbf{N}} C_n$, there exists a sequence $z_n \in C_n$ with $\|z_n - x\| \underset{n \in \mathbf{N}}{\to} 0$. Since $x_n = \pi(x_m|C_n)$ for $m > n$ and $z_n \in C_n$ we obtain

$$\|x_m - x_n\| \leqslant \|x_m - z_n\| \quad \text{if } m \geqslant n. \tag{1}$$

Let $\varepsilon > 0$ be fixed and $\delta = \delta(\varepsilon)$ be chosen according to our Remark with $r := \sup_{n \in \mathbf{N}}\|x_n\|$. As $\|z_n - x\| \underset{n \in \mathbf{N}}{\to} 0$, (1) implies that there exists $n_0(\varepsilon) \in \mathbf{N}$ such that

$$\|x_m - x_n\| \leqslant \|x_m - x\| + \delta/4 \quad \text{if } m > n \geqslant n_0(\varepsilon). \tag{2}$$

Since $\|x_m - x\| \underset{m \in \mathbf{N}_0}{\to} a$, according to (2) there exists $n_1(\varepsilon) \geqslant n_0(\varepsilon)$ such that

$$\|x_m - x_n\| \leqslant a + \delta/2 \quad \text{if } m > n \geqslant n_1(\varepsilon), m \in \mathbf{N}_0. \tag{3}$$

Since x_m, $m \in \mathbf{N}_0$, converges weakly to x, we have for each $n \in \mathbf{N}$ that $x_m - x_n$, $m \in \mathbf{N}_0$, converges weakly to $x - x_n$ and hence

$$\|x - x_n\| \leqslant \varliminf_{m \in \mathbf{N}_0} \|x_m - x_n\|. \tag{4}$$

Therefore for each $n \in \mathbf{N}$ there exists $k_n(\varepsilon) \in \mathbf{N}$ such that

$$\|x - x_n\| - \delta/4 \leqslant \|x_m - x_n\| \quad \text{if } m > k_n(\varepsilon), m \in \mathbf{N}_0. \tag{5}$$

Since $\|x - x_n\| \underset{n \in \mathbf{N}_0}{\to} a$, we obtain from (3) and (5) that there exists $n_2(\varepsilon) \geqslant n_1(\varepsilon)$ such that

$$a - \delta/2 \leqslant \|x_m - x_n\| \leqslant a + \delta/2$$
$$\text{if } m > n \geqslant n_2(\varepsilon), m > k_n(\varepsilon) \text{ and } m, n \in \mathbf{N}_0. \tag{6}$$

Now let $s_1, s_2 \in \mathbf{N}_0$ with $s_2 > s_1 > n_2(\varepsilon)$ and choose $m \in \mathbf{N}_0$ with $m > \max(n_2(\varepsilon), k_{s_1}(\varepsilon), k_{s_2}(\varepsilon))$. Then (6) implies that

$$\|x_m - x_{s_1}\| \leqslant \|x_m - x_{s_2}\| + \delta = d(x_m, C_{s_2}) + \delta. \tag{7}$$

Since $\|x_m - x_{s_2}\| = d(x_m, C_{s_2})$ and $x_{s_1}, x_{s_2} \in C_{s_2}$ we obtain from (7) according to our Remark that

$$\|x_{s_1} - x_{s_2}\| < \varepsilon \qquad \text{if } s_1, s_2 \in N_0, s_1, s_2 > n_2(\varepsilon).$$

This implies (∗). Hence $\|x_n - x\| \underset{n \in N_0}{\to} 0$, because x_n, $n \in N_0$, converges weakly to x. Therefore $x_n = \pi(x|C_n)$ for all $n \in N$, as $x_n = \pi(x_m|C_n)$, $n < m$, and the projection operator is continuous. Since x is in the strong closure of $\bigcup_{n \in N} C_n$, we immediately obtain that x_n, $n \in N$, converges strongly to x.

We remark that obviously the preceding result also holds true for martingales, indexed by a net.

Our theorem yields classical martingale representation theorems or Brunk's representation theorem and is moreover applicable to martingales of p-predictors or martingales of p-means.

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT ZU KÖLN, WEYERTAL 86-90, D-5000 KÖLN 41, FEDERAL REPUBLIC OF GERMANY

UNIVERSITÄT KONSTANZ, FACHBEREICH WIRTSCHAFTS-WISSENSCHAFTEN/STATISTIK, POSTFACH 7733, D-7750 KONSTANZ, FEDERAL REPUBLIC OF GERMANY