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# A SIMPLIFIED PROOF OF THE ERDŐS-FUCHS THEOREM

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ABSTRACT. We reprove the theorem of Erdős and Fuchs in additive number theory. Whereas their solution rested on some special results in the $L^2$ theory of Fourier series, ours avoids these.

We present a variant of the proof of the very pretty theorem of Erdős and Fuchs [1]. Our proof is technically a bit simpler than theirs but, of more importance, it has the aesthetic advantage of sticking closer to the spirit of generating functions.

THEOREM (ERDŐS-FUCHS). *Let $A$ be a set of nonnegative integers and denote by $r(n)$ the number of solutions to $n = a + a'$, $a, a' \in A$. If for some $C > 0$, $\sum_{k=0}^{n}(r(k) - C) = O(n^{\alpha})$, then $\alpha \geqslant \frac{1}{4}$.*

PROOF. If we write $A_n = \sum_{k=0}^{n}(r(k) - C)$ then we have

$$\left(\sum z^a\right)^2 = \frac{C}{1-z} + (1-z)\sum A_n z^n, \qquad A_n = O(n^{\alpha}), \tag{1}$$

Here, as later, we abbreviate our summation notation. It is to be generally understood that $a$ ranges over the set $A$, that $n$ ranges over the nonnegative integers, and, when we use the letter $b$, it will range over the nonnegative integers below $N$.

So let us multiply (1) by $(1 + z + z^2 + \cdots + z^{N-1})^2$, $N > 1$, and obtain thereby

$$\left(\sum z^a \sum z^b\right)^2 = \frac{C}{1-z}\left(\sum z^b\right)^2 + (1 - z^N)\sum z^b \cdot \sum A_n z^n \tag{2}$$

which in turn gives the inequality

$$\left|\sum z^a \sum z^b\right|^2 \leqslant \frac{CN^2}{|1-z|} + 2\left|\sum z^b \sum A_n z^n\right|. \tag{3}$$

We now integrate this inequality around the circle $|z| = r$, $r < 1$, with the measure of normalized arc length, i.e. $|dz|/2\pi r$.

First of all, setting $\sum z^a \cdot \sum z^b = \sum c_n z^n$, we observe that the $c_n$ are integers so that by Parseval's theorem we obtain

$$\int \left|\sum z^a \sum z^b\right|^2 = \sum c_n^2 r^{2n} \geqslant \sum c_n r^{2n} = \sum r^{2a} \cdot \sum r^{2b}. \tag{4}$$

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Furthermore if we assume, as we may, that $\alpha < 1$ then the relation (1) tells us that, as $z \to 1^-$, $(\Sigma z^a)^2 \sim C/(1-z)$ and this insures the existence of a $\gamma > 0$ for which $(\Sigma r^{2a})^2 > \gamma/(1-r^2)$. Also we have $\Sigma r^{2b} > N \cdot r^{2N}$ and so (4) leads to the inequality

$$\int \left| \sum z^a \sum z^b \right|^2 > \sqrt{\frac{\gamma}{1-r^2}} \cdot N r^{2N}. \tag{5}$$

As to the right-hand side of (3) we first recall the elementary inequality

$$\int \frac{1}{|1-z|} < 1 + \log \frac{1}{1-r^2} \tag{6}$$

(obtained, for example, from the expansion for $(1-z)^{-1/2}$). Next we apply Schwarz' inequality to deduce that

$$\left( \int \left| \sum z^b \sum A_n z^n \right| \right)^2 < \int \left| \sum z^b \right|^2 \cdot \int \left| \sum A_n z^n \right|^2$$
$$= \sum r^{2b} \cdot \sum A_n^2 r^{2n} < N \sum A_n^2 r^{2n}. \tag{7}$$

Now $A_n = O(n^\alpha)$, and so, because of the elementary inequality $\Sigma n^{2\alpha} r^{2n} = O(1/(1-r^2)^{2\alpha+1})$ (obtained, for example, by comparison with the definite integral), we may conclude from (7) that

$$2\int \left| \sum z^b \right| \left| \sum A_n z^n \right| < \frac{A\sqrt{N}}{(1-r^2)^{\alpha+1/2}}. \tag{8}$$

To complete the proof we choose $r^2 = 1 - N^{-\lambda}$, $\lambda > 1$, so that $r^{2N} > (1 - 1/N)^N > (1 - \frac{1}{2})^2 = \frac{1}{4}$ and hence combining (3), (5), (6) and (8) gives

$$(\sqrt{\gamma}/4) N^{\lambda/2+1} < CN^2(1 + \lambda \log N) + AN^{1/2 + \lambda\alpha + \lambda/2},$$

or

$$\sqrt{\gamma}/4 < CN^{1-\lambda/2}(1 + \lambda \log N) + AN^{\lambda\alpha - 1/2}. \tag{9}$$

If, finally, we assume $0 < \alpha < \frac{1}{4}$ then any choice of $\lambda$ which is above 2 but below $1/2\alpha$ (e.g. $\lambda = 3/(1+2\alpha)$) makes both terms on the right side of (9) go to 0 as $N \to \infty$ and this contradiction completes the proof.

## References

1. P. Erdős and W. H. J. Fuchs, *On a problem in additive number theory*, J. London Math. Soc. 31 (1956), 67–73.

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