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ON BOUNDED SETS IN INDUCTIVE LIMITS OF NORMED SPACES

KLAUS FLORET

ABSTRACT. A theorem on bounded sets in locally convex inductive limits is proven and applied in various special cases.

1. For many questions in analysis in locally convex spaces it is of crucial importance to know how the bounded sets behave. In particular in locally convex inductive limits of (injective) sequences of normed spaces $E = \operatorname{ind}_{n\to} E_n$ there are various examples given by B. M. Makarov **[12]** with the property that bounded sets in E are not situated in any E_n, or are not bounded there. It is the purpose of this note to give a sufficient condition for a sequence (E_n) of normed spaces to generate regularly the inductive limit $E = \operatorname{ind}_{n\to} E_n$, i.e., every bounded set in E is situated in some E_{n_0} and bounded there. Since the closure of $\{0\}$ in E is bounded in E, regular inductive limits are Hausdorff; this observation yields an easy proof and a slight improvement (see §7) of a frequently used result of E. Dubinsky **[3]** for an inductive limit to be Hausdorff.

2. The terminology is that of G. Köthe **[10]** and **[6]**. The inductive sequences

$$E_1 \xrightarrow{\pi_1} E_2 \xrightarrow{\pi_2} E_3 \xrightarrow{\pi_3} \cdots$$

consist of separated locally convex spaces E_n and injective (continuous) linking mappings $\pi_n: E_n \to E_{n+1}$; this means that the generating spaces can be considered as linear subspaces of $E = \operatorname{ind}_{n\to} E_n$, the linking mappings now being simply the inclusion mappings. Denoting by τ_n the topology of E_n and by τ the (locally convex) inductive limit topology of $E = \bigcup_{n=1}^{\infty} E_n$, the sequence (E_n) is called:

(a) *α-regular*, if every τ-bounded subset of E is situated in some E_n,

(b) *β-regular*, if every τ-bounded subset of E, which is situated in some E_n, is τ_m-bounded for some $m \in \mathbf{N}$,

(c) *regular*, if it is both α- and β-regular, i.e. for every τ-bounded subset $A \subset E$ there is an $n \in \mathbf{N}$ such that $A \subset E$ and A is τ_n-bounded.

The mentioned counterexamples of B. M. Makarov are (LB)-spaces which show that (a) $\not\to$ (b), (b) $\not\to$ (a) and that an (LB)-space can be neither α- nor

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β-regular (taking for example the direct sum of the former counterexamples).

It is obvious that the regularity properties of a sequence are inherited by equivalent sequences. The inductive limit of a regular sequence of different metrizable spaces is never metrizable [5, p. 94].

3. Grothendieck's factorization theorem for (LF)-spaces [7, I, p. 16] implies that a *Hausdorff* (LF)-space is regular if and only if it is Mackey-complete. But both, separatedness and completeness are not always easy to check.

THEOREM. *If (E_n) is an inductive sequence of normed spaces E_n with closed unit balls K_n, such that, for all sequences (ε_m) of positive numbers and all $n \in \mathbf{N}$, $\sum_{m=1}^n \varepsilon_m K_m$ is closed in E_{n+1}, then $E = \operatorname{ind}_{n\to} E_n$ is regular, in particular Hausdorff.*

PROOF. The method is more or less standard. Let $A \subset E$ be τ-bounded and not τ_n-bounded for all n.

(a) Assume that there are $\varepsilon_m > 0$ and $x_m \in A$ with

$$\frac{1}{m} x_m \notin \sum_{i=1}^{n} \varepsilon_i K_i =: U_n, \qquad m = 1, 2, \ldots, n. \tag{$*$}$$

Then, because $U_{n-1} \subset U_n$, $(*)$ holds for all $m,n \in N$:

$$\frac{1}{m} x_m \notin U := \bigcup_{n=1}^{\infty} U_n, \qquad m = 1, 2, \ldots.$$

But U is a τ-neighbourhood of zero and $((1/m)x_m)$ τ-converges to zero (A being τ-bounded)–a contradiction is established.

(b) Thus (ε_m) and (x_m) with $(*)$ have to be constructed: $A \not\subset K_1$, so there is an $x_1 \notin K_1$, $\varepsilon_1 := 1$. Proceeding by induction, assume that $\varepsilon_m > 0$ and $x_m \in A$ with

$$\frac{1}{m} x_m \notin \sum_{i=1}^{n} \varepsilon_i K_i =: U_n, \qquad m \leqslant n,$$

are given. By the assumption of the theorem U_n is closed in E_{n+1}; thus there is an ε_{n+1} with

$$(1/m)x_m \notin U_n + \varepsilon_{n+1} K_{n+1} =: U_{n+1}, \qquad m \leqslant n$$

(the x_m need not be elements of E_{n+1}). Since U_{n+1} is bounded in E_{n+1}, the set A is not contained in $(n+1)U_{n+1}$ and this implies the existence of an $x_{n+1} \in A$ with

$$\frac{1}{n+1} x_{n+1} \notin U_{n+1} = \sum_{i=1}^{n+1} \varepsilon_i K_i. \quad \square$$

4. It is clear that the assumption of the theorem must only be satisfied for, say, rapidly decreasing sequences (ε_n). But the summation of closed sets is always problematic which is illustrated by a theorem of V. Klee [**10**, §24.4] which states that a quasi-complete locally convex space is semi-reflexive if and only if the sum of each two convex, bounded, closed sets is closed (see §8 for an example).

COROLLARY 1. *Let* (E_n) *be an inductive sequence of normed spaces,* F *a semireflexive locally convex space and* T *an injective, continuous operator* $E = \mathrm{ind}_{n\to} E_n \to F$ *such that* TK_n *is closed for all* n. *Then* $E = \mathrm{ind}_{n\to} E_n$ *is regular.*

PROOF. All TK_n are $\sigma(F, F')$-compact and so is

$$\sum_{m=1}^{n} \varepsilon_m TK_m = T\left(\sum_{m=1}^{n} \varepsilon_m K_m \right),$$

in particular closed: T being injective and continuous yields that the theorem applies. □

Take for example Köthe echelon-spaces: The E_n are of the form (ω denotes the Fréchet-Schwartz space of all real or complex sequences, $a_k > 0$)

$$l^1(a_k) := \left\{ (\xi_k) \in \omega \middle| \|(\xi_k)\| := \sum_{k=1}^{\infty} |\xi_k| a_k < \infty \right\}.$$

The unit ball of $l^1(a_k)$ is closed and bounded in ω, so Corollary 1 gives:

If $0 < a_k^{n+1} \leqslant a_k^n$ for all $k,n \in \mathbf{N}$ then the space $\mathrm{ind}_{n\to} l^1(a_k^n)$ is regular.

This is a result of E. Dubinsky [**2**, Proposition 2].

Injective, continuous operators $T\colon G \to F$ between Banach spaces which map the closed unit ball onto a closed set are studied by H. P. Lotz, N. T. Peck, and H. Porta [**11**] and are called by them semi-embeddings.

5. Another use of the fact that the sum of compact sets is compact yields

COROLLARY 2. *The inductive limit of a sequence of dual Banach spaces such that the inclusion mappings are dual mappings is regular.*

PROOF. With $E_n = G_n'$, the unit ball (in the dual norm) in E_n is $\sigma(E_n, G_n)$-compact and the inclusion map is $\sigma(E_n, G_n) - \sigma(E_{n+1}, G_{n+1})$ continuous. □

The unit ball in an equivalent norm of a dual Banach space G' need not be $\sigma(G', G)$-compact: It is easily checked that

$$e = (1, 1, \dots) \notin K := \{(\xi_n) \in l^\infty | \max(\|(\xi_n)\|_\infty, 2 \limsup |\xi_n|) \leqslant 1\}$$

but e is in the $\sigma(l^\infty, l^1)$-closure of K (this norm is from an example in [**11**]). The existence of such a norm in the dual characterizes nonreflexive Banach spaces [**8**, p. 155].

Corollary 2 can also be proven by the observation that $\mathrm{proj}_{\leftarrow n} G_n'$ is quasibarrelled (strongly bounded sets in the dual are equicontinuous).

6. The familiar fact that inductive limits of sequences of locally convex spaces with (weakly) compact linking mappings are regular [**4**], [**9**] is a consequence of Corollary 2, using, for example, the following known lemma

[7, pp. 104–105]; [11]. (For a bounded, absolutely convex subset A of a locally convex space, $[\![A]\!]$ denotes the linear hull of A equipped with the Minkowski norm m_A.)

LEMMA. *Let K be a absolutely convex, weakly compact subset of a locally convex space E, then there is a Banach-space B such that $[\![K]\!] = B'$ isometrically and $[\![K]\!] = B' \hookrightarrow E$ is $\sigma(B', B) - \sigma(E, E')$ continuous.*

(Take $B = E'/\ker m_{K^0}$ with the quotient norm of m_{K^0} and notice that the unit ball of B' in E'' is $K^{00} = K$.)

7. E. Dubinsky [3] gave a sufficient condition for an inductive limit to be Hausdorff which turned out being very useful. The present method gives

COROLLARY 3. *If (E_n) is an inductive sequence of normed spaces such that the bidual mappings $E_n'' \to E_{n+1}''$ are injective, then $E = \operatorname{ind}_{n\to} E_n$ is Hausdorff and β-regular.*

PROOF. $\hat{E} := \operatorname{ind}_{n\to} E_n''$ is regular by Corollary 2, in particular Hausdorff; since the map $E = \operatorname{ind} E_n \hookrightarrow \hat{E}$ is continuous and injective this implies, that E is also Hausdorff. A bounded set $A \subset E$ is bounded in $\hat{E}$ and thus situated and bounded in some E_n'': If A is in E_m then it is bounded in E_k, $k = \max(m, n)$, since $E_k \subset E_k''$ as a subspace. □

In particular all inductive limits (definitions similar to that of $l^1(a_k)$) $\operatorname{ind}_{n\to} c_0(a_k^n)$ are Hausdorff and β-regular because the bidual of $c_0(a_k) \hookrightarrow c_0(b_k)$ is $l^\infty(a_k) \hookrightarrow l^\infty(b_k)$, E. Dubinsky (unpublished) constructed such a sequence of c_0's which is not regular.

8. The difficulty applying the theorem comes from V. Klee's characterization of nonreflexive spaces. That the closed sets with a nonclosed sum are not all pathological is shown with the following example. It shows also drastically why the theorem does not help in the case of strict embeddings (i.e., E_n being a subspace of E_{n+1})–but this is not severe because there is a very general result available (e.g. [6, p. 127]).

There is a (necessarily nonreflexive) Banach-space with a closed hyperplane H, K the unit ball, such that $K + K \cap H$ is not closed.

PROOF. Consider in c_0, with the sup-norm $\| \ \|_\infty$, the closed hyperplane

$$H := \left\{ (\xi_n)_{n=1}^{\infty} \in c_0 \,\middle|\, \sum_{n=1}^{\infty} \xi_n \frac{1}{2^n} = 0 \right\}.$$

(a) If $(\xi_n) \in K \cap H$ then $\xi_1 < 1$, since otherwise

$$\frac{1}{2} = \sum_{n=2}^{\infty} (-\xi_n) \frac{1}{2^n}$$

which is impossible because of $|\xi_n| \leq 1$ and $\xi_n \to 0$.

(b) This implies $2e_1 := (2, 0, 0, \dots) \notin K + K \cap H$.

(c) If $0 < \varepsilon < 1$ then there is an $x_\varepsilon = (1 - \varepsilon, \dots) \in K \cap H$: Take

$$d = \sum_{n=2}^{N} \frac{1}{2^n} > \frac{1-\varepsilon}{2}$$

and

$$x_\varepsilon = \left(1 - \varepsilon, \frac{\varepsilon - 1}{2d}, \ldots, \frac{\varepsilon - 1}{2d}, 0, \ldots\right),$$

the zeros starting in component $N + 1$.

(d) Since $(2 - \varepsilon)e_1 - x_\varepsilon \in K$ it follows that

$$2e_1 = \lim_{\varepsilon \to 0}\left[(2 - \varepsilon)e_1 - x_\varepsilon + x_\varepsilon\right] \in \overline{K + K \cap H}.$$

This, together with (b) establishes the desired example. □

Does the existence of such a hyperplane characterize nonreflexive Banach-spaces? Using the theorem of R. C. James on the supremum of linear functionals, more or less the same proof as above shows that this is true up to a slight modification of the norm.

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Department of Mathematics, State University of New York, Buffalo, New York 14214

Mathematisches Seminar der Universität, D-23 Kiel, Bundesrepublik Deutschland (Current address)