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FACTORIZATIONS OF FREE ACTIONS OF FINITE GROUPS ON COMPACT Q-MANIFOLDS

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ABSTRACT. We show that every free action of a finite group G on a compact Q-manifold $K^k \times Q$ is conjugate to the product of a G-free action on a regular neighborhood of K in $\mathbf{R}^{2k+1}(\mathbf{R}^7$, if $k = 2)$ and the identity on Q. Also, sharper results for special finite complexes K are obtained.

1. Free action of finite cyclic groups on $S^n \times Q$. Two free actions of a group G on a topological space M are conjugate if there is a homeomorphism f of M onto itself such that $f \circ \lambda_g \circ f^{-1} = \lambda'_g$, where λ_g and λ'_g are correspondent homeomorphisms of M defined by any element g of G.

One wonders which free action of a finite group G on $S^n \times Q$ is conjugate to the product of a G-free action on the sphere S^n and the identity on the Hilbert cube Q; or equivalently, are their orbit spaces homeomorphic? By use of Chapman [$\mathbf{Ch}_1$] and West [$\mathbf{We}_1$] these are equivalent to showing that the orbit space of the given free action, compact Q-manifold, is simple homotopy equivalent to a compact topological n-manifold without boundary.

In this section, we will prove the following theorems.

THEOREM 1. *Every $\mathbf{Z}_2$-free action on $S^n \times Q$ is conjugate to the product of the standard $\mathbf{Z}_2$-free action on S^n and the identity on Q.*

THEOREM 2. *For $p > 2$, every $\mathbf{Z}_p$-free action on $S^{2n-1} \times Q$ is conjugate to the product of a $\mathbf{Z}_p$-free action on S^{2n-1} ($S^3 \times I^2$, if $n = 2$) and the identity on Q.*

It is interesting to mention that a similar statement is not true for arbitrary finite groups.

Let N be a Poincaré complex whose fundamental group is the symmetric group S_3 and whose universal covering space $\tilde{N}$ has the homotopy type of the 3-sphere S^3. (Such a complex exists by the second part of Theorem 4.3 in [$\mathbf{W}_1$].) Moreover, since the obstruction to N being of finite type lies in $\tilde{K}^0(S_3) = 0$ [$\mathbf{W}_2$, p. 67], we may assume that N is a finite complex.

Now, Theorem 5.2 of [**We**] or Lemma 4.2 of [$\mathbf{Ch}_2$] shows that $\tilde{N} \times Q \approx S^3 \times Q$. Therefore, an exotic free action of S_3 on $S^3 \times Q$ is obtained naturally,

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since S_3 cannot act freely on S^3 ([M]). For large n, exotic free actions of S_3 on $S^n \times Q$ can also be similarly obtained.

For the notion of finite Poincaré complex, we refer to Chapter 2 of [W_3].

PROOF OF THEOREM 2. For $n = 1$, it is trivial. So, we assume $n > 2$.

Step 1. *The orbit space* $(S^{2n-1} \times Q)/\mathbf{Z}_p$ *is homotopy equivalent to a lens space* $L_n(p; q, 1, \ldots, 1)$ *for some* q.

Let N be a finite complex such that $N \times Q \approx (S^{2n-1} \times Q)/\mathbf{Z}_p$. Then N is a $(\mathbf{Z}_p, 2n-1)$-polarisation (Definition 2.1 of [T-W]), and by Remark 2.6 of [T-W], it is a finite Poincaré complex.

Now, by Theorem 4E.3 of [W_3], we may assume that $N = K \cup_f e^{2n-1}$, where K is the $(2n-2)$-skeleton of the lens space $L_n(p; 1, \ldots, 1)$. Hence, the fibration $\mathbf{Z}_p \to \tilde{N} \to N$ is classified by a map φ: $N \to L_n(p; 1, \ldots, 1)$. (For notation of lens space refer to [Co_1].)

It is clear that the induced map $\varphi_{\#}$ on fundamental groups is an isomorphism. We claim that the degree a of φ is relatively prime to p. Indeed, let α be a generator of $\pi_1(N) = \mathbf{Z}_p$. By use of the cell-structure of $L_n(p; 1, \ldots, 1)$ (refer to [Co_1]), we can define skeleton-wise a map ψ from $L_n(p; 1, \ldots, 1)$ to N such that $\psi_{\#}(\varphi_{\#}(\alpha)) = \alpha$; hence, $(\varphi \circ \psi)_{\#}$ is the identity on $\pi_1(L_n(p; 1, \ldots, 1))$. Now, (29.4) of [$Co_1$] shows that the degree of $\varphi \circ \psi \equiv 1 \pmod{p}$. So, the claim is proved.

Again, by (29.4) of [Co_1], we have a map θ: $L_n(p; 1, \ldots, 1) \to L_n(p; q, 1, \ldots, 1)$ of degree q inducing an isomorphism on fundamental groups, where q is an integer such that $aq \equiv 1 \pmod{p}$.

Now, using the cell-structure of $N = K \cup_f e^{2n-1}$ to modify the map $\theta \circ \varphi$ over a closed ball in Int e^{2n-1} (see [Co_1, p. 95]), we may obtain a map μ: $N \to L_n(p; q, 1, \ldots, 1)$ such that

(a) $\mu_{\#}$ is an isomorphism on fundamental groups,

(b) degree $\mu = 1$.

Then, it is routine to show that μ is a homotopy equivalence. The proof of Step 1 is complete.

Step 2. *Case* 1. $(2n - 1 > 3)$.

Let $(W; L_n(p; q, 1, \ldots, 1), M)$ be a PL h-cobordism such that the torsion $\tau(W, L_n(p; q, 1, \ldots, 1)) = \tau(\mu^{-1})$, the torsion of the homotopy equivalence μ^{-1} in Step 1 [H, Theorem 12.1]. Then N and M have the same simple homotopy type. The proof of this case is complete.

Case 2. $(2n - 1 = 3)$. Let $(W^6; L_2(p; q, 1) \times I^2, M^5)$ be a suitable h-cobordism as in Case 1 such that $\partial W = [L_2(p; q, 1) \times I^2] \cup M^5 \cup [L_2(p; q, 1) \times \partial I^2 \times I]$. In particular $\partial M = L_2(p; q, 1) \times \partial I^2$.

Now, it is clear that $(\tilde{W}^6; \tilde{L}_2(p; q, 1) \times I^2, \tilde{M}^5)$ is also an h-cobordism (it is trivial on $\tilde{L}_2(p; q, 1) \times S^1 \times I$). Moreover, since $\pi_1(\tilde{W}^6) = 0$, we have, by the relative h-cobordism theorem ([R-S, Theorem 6.18])

$$\tilde{M}^5 \approx \tilde{L}_2(p; q, 1) \times I^2 = S^3 \times I^2.$$

Therefore, the proof of Theorem 2 is complete. □

PROOF OF THEOREM 1. Using notation of Step 1 above, N is also a finite Poincaré complex of formal dimension n, by Remark 2.6 of [T-W]. Then, $N \simeq P_n(\mathbf{R})$ by use of (4.3) of [W_3] or by going along the proof of Step 1 above.

Finally, the proof of the theorem is complete since the Whitehead group of $\mathbf{Z}_2$ is trivial [Co_1]. □

2. Free actions of a finite group G on compact Q-manifolds. For compact ANR's X and Y, let $X \simeq Y$ ($X \simeq_s Y$) mean that X and Y have the same (simple) homotopy type.

Given a simplicial complex X and a normal subgroup H of $\pi_1(X, x_0)$ such that $G \approx \pi_1(X, x_0)/H$ is of finite order, denote by $(X(G), e_0)$ the G-covering space of (X, x_0); i.e. the transformation group of the covering map $(X(G), e_0) \underset{p}{\to} (X, x_0)$ is G, and $p_*\pi_1(X(G), e_0) = H$. It is well known that $X(G)$ is unique up to homeomorphism (see [G, Corollary 6.9]).

Let $\tilde{X}$ be the universal covering space of a finite simplicial complex X, Λ the integral group ring $\mathbf{Z}\pi_1 X$ and $C_*(X)$ the cellular chain complex of $\tilde{X}$. As in [W_3], we let

$$H_*^t(X; B) = H\big(C_*(X) \otimes_\Lambda B\big),$$

$$H^*(X; B) = H\big(\mathrm{Hom}\big(C_*(X); B\big)\big),$$

where B is a (right) Λ-module.

According to the proof of Theorem F in [W_2], the condition D_n can be restated as follows:

(i) $H_i(\tilde{X}; \mathbf{Z}) = 0$, $i > n$,

(ii) $H^{n+1}(X; B) = 0$ for every Λ-module B.

For an integer $k > 1$, we define the integer $q(k)$ as follows

(i) $q(k) = 3$ if $k = 2$,

(ii) $q(k) = k$ if $k \neq 2$.

The main result of this section is the following theorem.

THEOREM 3. *Given a finite complex K^k, any G-free action on the compact Q-manifold $K \times Q$ is conjugate to the product of a G-free action on a regular neighborhood of K in $\mathbf{R}^{2q(k)+1}$ and the identity on Q.*

As we mentioned before, there might be exotic free actions of finite groups on $S^k \times Q$. However, since the regular neighborhood of S^k in $\mathbf{R}^m$ is homeomorphic to $S^k \times I^{m-k}$, we have the following factorization.

COROLLARY. *Any G-free action on $S^k \times Q$ is conjugate to the product of a G-free action on $S^k \times I^{q(k)}$ and the identity on Q.*

Before giving a proof for Theorem 3, we need some technical lemmas.

LEMMA 1. *Let G act freely and simplicially on a finite complex Y^q and the orbit space X^q. If N is a regular neighborhood of X in $\mathbf{R}^{2q+1}$, then $N(G)$ is a regular neighborhood of Y in $\mathbf{R}^{2q+1}$.*

PROOF. Let p: $N(G) \to N$ be the orbit map, then it is obvious that p is a PL-immersion. Approximate p by $\bar{p}$ with $\bar{p}$ being in general position. Then, $\bar{p}$ is also a PL-immersion and $\bar{p}|Y$ is a PL-embedding. So there is a small regular neighborhood $U(G)$ of Y in $N(G)$ such that $\bar{p}|U(G)$ is a PL-embedding, where U is a small regular neighborhood of X in N. So $U(G) \approx N(G)$ by the uniqueness of regular neighborhoods.

LEMMA 2. *Let G act freely on a compact ANR X. If X has the homotopy type of a finite complex K^k, then the orbit spaces Y has the homotopy type of a finite complex of dimension $q(k)$.*

PROOF. It is obvious that Y is a compact ANR. Let f: $Y' \to Y$ be a homotopy equivalence, where Y' is a finite complex [$\mathbf{We_2}$]. Then consider the orbit map p: $Y'(G) \to Y'$ induced by f. So, we may assume that X is a finite complex and the action is simplicial.

Now, we intend to use Theorem F of [$\mathbf{W_2}$]. Since Y is a finite complex, it suffices to show that Y satisfies the condition D_n: $H_i(\tilde{Y}) = 0$ for $i > q(k)$ and $H^{q(k)+1}(Y; B) = 0$ for every $Z\pi_1 Y$-module B.

By Lemma 1 above, we may assume that X is a regular neighborhood of a copy of K in $\mathbf{R}^{2q(k)+1}$; so $(Y, \partial Y)$ is a finite Poincaré complex of formal dimension $2q(k) + 1$.

By general position theorem, it follows that the pair $(X, \partial X)$ is $q(k)$-connected. Then, so is the pair $(Y, \partial Y)$ by the five-lemma and the homotopy lifting property of covering maps.

Now by Lemma 1.1 of [$\mathbf{W_2}$], the pair $(Y, \partial Y)$ is homotopy equivalent to a pair $(Y_1, \partial Y)$ rel ∂Y such that $Y_1 - \partial Y$ contains no cells of dimension less than $q(k) + 1$. So, it follows that $H_r^l(Y, \partial Y; B) = 0$ and $H^r(Y, \partial Y; B) = 0$ for all $r \leqslant q(k)$ and $Z\pi_1 Y$-module B.

Therefore, the Poincaré duality theorem in [$\mathbf{W_3}$] shows that $H^r(Y; B) = 0$ and $H_r^l(Y; B) = 0$ for all $r \geqslant q(k) + 1$ [$\mathbf{W_3}$, Theorem 2.1].

The proof of Lemma 2 is complete.

PROOF OF THEOREM 3. Chapman [$\mathbf{Ch_1}$] shows that $(K \times Q)/G \approx L \times Q$ for some finite complex L. We may assume that $\dim L \leqslant q(k)$. For $L \simeq L'$, $\dim L' \leqslant q(k)$ by Lemma 2 above. If $q(k) = 1$, then $L \simeq_s L'$ since the Whitehead groups of free groups are trivial [$\mathbf{Co_1}$, Theorem 11.6]. If $q(k) \geqslant 3$, from Lemma 1.1 of [$\mathbf{Co_2}$] it follows that, given any torsion $\tau \in \mathrm{Wh}(\pi_1 L')$, there is a finite complex $L'' \supset L'$ such that $\dim(L'' - L') \leqslant 3$ and the torsion of the pair (L'', L') is τ. So, there exists a finite complex L'' of dimension $q(k)$ such that $L'' \simeq_s L$.

Let N be a regular neighborhood of $L(G)$ in $\mathbf{R}^{2q(k)+1}$ given by Lemma 1. Then, it is easy to show that $K \simeq_s N$.

Now, let f: $K \to \mathrm{Int}\, N \subset N$ be a PL-embedding which defines the simple homotopy equivalence. Let W be a regular neighborhood of $f(K)$ in Int N, then $(N - \mathrm{Int}\, W; \partial W, \partial N)$ is an h-cobordism of zero torsion. For it is routine to show that the inclusion maps $\partial N \to (N - \text{Int } W)$ and $\partial W \to (N - \text{Int } W)$ are homotopy equivalences; and, furthermore, $\partial W \to (N - \text{Int } W)$ is a simple homotopy equivalence by use of the excision theorem 20.3 in [Co_1] and the functorial property of $G \to \text{Wh}(G)$ (see [Co_1, p. 40]).

Therefore, from the h-cobordism theorem it follows easily that $N \approx W$. The proof of the theorem is complete. □

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