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A COUNTABLY COMPACT, FIRST COUNTABLE, NONNORMAL T_2-SPACE

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ABSTRACT. We construct a space which has the properties listed in the title and which is also locally compact, locally countable, zero-dimensional, ω-bounded and has cardinality $\aleph_1$.

1. Introduction. The fact that every countably compact, first countable, T_2-space is regular is a result which is well known and easy to prove. It is a natural question to ask if every such space is also normal. We first heard of this question in 1969 when it was raised in a conversation by R. M. Stephenson, Jr. He sent this question to Mary Ellen Rudin, and she sent back an example of a countably compact, first countable, nonnormal T_2-space. Since her example has never been published, we take this opportunity to mention that its construction was based on the techniques in [4], and in particular, the construction made use of the continuum hypothesis (CH). Professor Rudin conjectured in her letter that such an example could be constructed without using any set-theoretic assumptions beyond the usual axioms of set-theory including the axiom of choice (ZFC). The example which we construct in this paper shows that her conjecture is correct.

Several other countably compact, first countable, nonnormal T_2-spaces have been constructed recently. A. Ostaszewski (using the set-theoretic assumption $\diamond$) [3] constructed such a space which is, in addition, hereditarily separable. M. Wage (also using $\diamond$) [5] constructed one which is hereditarily separable and perfect. M. Dahroug (using Martin's Axiom plus not CH) constructed one which is separable (unpublished), and W. Weiss also constructed separable ones under several other set-theoretic assumptions (unpublished). It should be noted that Rudin's example (using CH) is separable.

In this paper, we construct within ZFC a space having the properties listed in the title and abstract. Since this space is ω-bounded (i.e., every countable subset has compact closure) it is not separable. It follows from the preceding discussion that a natural question to raise now is this: Does there exist within ZFC a countably compact, first countable, nonnormal, separable, T_2-space?

2. The example. We first establish some notation. Let ω denote the set of finite ordinals and let ω_1 denote the set of countable ordinals. Put

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$$\Lambda = \{\lambda < \omega_1 : \lambda \text{ is a limit ordinal}\},$$
$$A = \Lambda \cup \{\lambda + 2n : \lambda \in \Lambda \cup \{0\} \text{ and } 1 \leqslant n < \omega\}, \quad \text{and}$$
$$B = \Lambda \cup \{0\} \cup \{\lambda + 2n - 1 : \lambda \in \Lambda \cup \{0\} \text{ and } 1 \leqslant n < \omega\}.$$

The set X on which we will define the desired topology is the subset of $\omega_1 \times (\omega + 2)$ defined by $X = (\omega_1 \times \omega) \cup H \cup K$ where $H = \{h_\alpha = (\alpha, \omega): \alpha \in A\}$ and $K = \{k_\alpha = (\alpha, \omega + 1): \alpha \in B\}$. The sets H and K will be used to show that the space is not normal.

The construction of the topology T on X is by transfinite induction, and is indebted to the constructions of Ostaszewski and Wage mentioned in §1. The main successor ordinal step is given in the following result.

LEMMA. *Let Z be a set, and S a T_2-topology on Z which has a countable base consisting of compact-open sets. Let H, K be disjoint closed subsets of Z, and p, q two points not in Z. Then there exists a topology T on $Y = Z \cup \{p\} \cup \{q\}$ such that*

(a) *(Z, S) is an open subspace of (Y, T);*

(b) *(Y, T) is a compact, T_2-space having a countable base of compact-open sets;*

(c) *$H \cup \{p\}$ and $K \cup \{q\}$ are (disjoint) closed sets in (Y, T).*

PROOF. Since Z has a countable base and is zero dimensional, we have $\text{Ind}(Z) = 0$ [**1**, Chapter II, §2.E.]. Thus, there exist disjoint clopen sets U, V in Z such that $Z = U \cup V$, $H \subset U$, and $K \subset V$. Let $\{B_i: i < \omega\}$ be a countable base of compact-open sets of Z. Define for $n < \omega$

$$V(p, n) = \{p\} \cup \Big(U \setminus \bigcup_{i \leqslant n} B_i\Big), \quad \text{and}$$
$$V(q, n) = \{q\} \cup \Big(V \setminus \bigcup_{i \leqslant n} B_i\Big).$$

Let T be the topology on Y having as a base $S \cup \{V(p, n): n < \omega\} \cup \{V(q, n): n < \omega\}$. The space (Y, T) is a two-point compactification of (Z, S) satisfying the properties (a), (b) and (c). $\square$

Construction of the example. For $\alpha < \omega_1$, put

$$H_\alpha = \{h_\beta: \beta \in A \text{ and } \beta < \alpha\},$$
$$K_\alpha = \{k_\beta: \beta \in B \text{ and } \beta < \alpha\}, \quad \text{and}$$
$$X_\alpha = (\alpha \times \omega) \cup H_\alpha \cup K_\alpha.$$

Assume for $\alpha < \gamma$ (where $\gamma < \omega_1$) we have defined topologies T_α on X_α such that the following hold for all $\beta < \alpha < \gamma$.

(1) (X_β, T_β) is a open subspace of (X_α, T_α).

(2) (X_α, T_α) is a T_2-space having a countable base of compact-open sets, and further if α is a successor ordinal, then (X_α, T_α) is compact.

(3) H_α and K_α are closed (disjoint) sets in (X_α, T_α).

(4) If $\lambda + 2n < \alpha$ and λ is a limit ordinal or $\lambda = 0$ and $1 \leqslant n < \omega$, then the

sequence $\{(\lambda + 2n, i): i < \omega\}$ converges to $h_{\lambda+2n}$ in (X_α, T_α) and if $\lambda + 2n - 1 < \alpha$, then the sequence $\{(\lambda + 2n - 1, i): i < \omega\}$ converges to $k_{\lambda+2n-1}$ in (X_α, T_α).

(5) $(\alpha \times \omega)$ with the product topology (where α and ω have the order topology) is an open subspace of (X_α, T_α).

We construct (X_γ, T_γ).

Case 1. γ is a limit ordinal. Put $T_\gamma = \{U \subset X_\gamma : U \cap X_\alpha \in T_\alpha \text{ for all } \alpha < \gamma\}$. Then (X_γ, T_γ) is the direct limit of the subspaces (X_α, T_γ), and (1)–(5) are easily verified for $\alpha = \gamma$.

Case 2. $\gamma = \lambda + 1$ where λ is a limit ordinal.

For each (λ, i) where $i < \omega$ and for each $\beta < \lambda$, define $V((\lambda, i), \beta) = (\beta, \lambda] \times \{i\}$. In order to define T_γ, we define an intermediate space (Y, S) as follows. Set $Y = X_\lambda \cup \{(\lambda, i): i < \omega\}$ and let S be the topology on Y having as a base $S \cup \{V((\lambda, i), \beta): i < \omega, \beta < \lambda\}$. Since $(\lambda \times \omega)$ is an open subspace of (X_λ, T_λ), we see that each $V((\lambda, i), \beta) \cap X_\lambda = (\beta, \lambda) \times \{i\} \in T_\lambda$. Thus (X_λ, T_λ) and $(\lambda + 1) \times \omega$ are open subspaces of (Y, S) and H_λ and K_λ are closed sets in (Y, S). Now $X_\gamma = Y \cup \{h_\lambda\} \cup \{k_\lambda\}$. Let T_γ be the topology on X_γ given by Lemma 1 where $H = H_\lambda$, $K = K_\lambda$, $p = h_\lambda$ and $q = k_\lambda$. So (X_γ, T_γ) is a compact T_2-space having a countable base of compact-open sets and H_γ, K_γ are closed (disjoint) in (X_γ, T_γ). Thus (1), (2), (3) and (5) hold, and there is nothing new to check for (4).

Case 3. $\gamma = \lambda + 2n$ where λ is a limit ordinal and $1 \leqslant n < \omega$.

Note that $X_\gamma \setminus X_{\gamma-1} = \{(\lambda + 2n - 1, i): i < \omega\} \cup \{k_{\lambda+2n-1}\}$. Thus we need only define neighborhoods for these points. Define T_γ so that $\{(\lambda + 2n - 1, i): i < \omega\}$ is a sequence of isolated points which converges to $k_{\lambda+2n-1}$. To do this put $V(k_{\lambda+2n-1}, i) = \{k_{\lambda+2n-1}\} \cup \{(\lambda + 2n - 1, j): i < j < \omega\}$ for all $i < \omega$, and let T_γ be the topology on X_γ having as a base $T_{\lambda+2n-1} \cup \{V(k_{\lambda+2n-1}, i): i < \omega\} \cup \{(\lambda + 2n - 1, i): i < \omega\}$. Properties (1)–(5) are easy to check for $\alpha = \gamma$ because (X_γ, T_γ) is the disjoint union of $(X_{\gamma-1}, T_{\gamma-1})$ and a convergent sequence. (Also note that $h_{\lambda+2n-1}$ is not defined; so there is nothing to do for H at this step.)

Case 4. $\gamma = \lambda + 2n + 1$ where $1 \leqslant n < \omega$. Proceed as in Case 3 except make $\{(\lambda + 2n, i): i < \omega\}$ a sequence of isolated points which converges to $h_{\lambda+2n}$.

Define T_{ω_1} as in the limit ordinal case. Then $T = T_{\omega_1}$ is a topology on X. We check that the space (X, T) has the desired properties. Since $\bigcup\{T_\alpha: \alpha < \omega_1\}$ is a base for T, it is clear that (X, T) is a first countable zero dimensional T_2-space. For each $\alpha < \omega_1$, $X_{\alpha+1}$ is a compact-open subspace of X. This implies that X is locally compact, locally countable and ω-bounded (since each countable subset of X is contained in some $X_{\alpha+1}$).

To show that X is not normal it suffices to show that if U and V are open sets in X such that $U \supset H$ and $V \supset K$ then $\overline{U} \cap \overline{V} \neq \varnothing$.

Since $\{(\lambda + 2n, i): i < \omega\}$ converges to $h_{\lambda+2n}$ for all limit ordinals λ, there exist $n_0 < \omega$ and an uncountable set $F_0 \subset \omega_1$ such that if $\alpha \in F_0$ and $i \geqslant n_0$ then $(\alpha, i) \in U$. Similarly for V, there exist $n_1 < \omega$ and an uncountable set $F_1 \subset \omega_1$ such that if $\alpha \in F_1$ and $i \geqslant n_1$, then $(\alpha, i) \in V$. Pick a sequence of countable ordinals from $F_0 \cup F_1$ such that $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$, and such that $\alpha_i \in F_0$ if and only if i is an even integer. Define $\alpha = \sup\{\alpha_i: i < \omega\}$, and $n = \max\{n_0, n_1\}$. Then $(\alpha, n) \in \overline{U} \cap \overline{V}$ since (α, n) has its usual neighborhoods in $\omega_1 \times \omega$ by property (5). This shows that (X, T) is not normal.

REMARK. If we apply the construction of F. B. Jones [2] to the space (X, T) we get a countably compact, first countable, T_2-space which is not completely regular.

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