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MONOTONE MAPS OF HEREDITARILY INDECOMPOSABLE CONTINUA

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ABSTRACT. We prove that every hereditarily indecomposable continuum is the image under an open, monotone map of a one-dimensional hereditarily indecomposable continuum. Thus there exists a one-dimensional hereditarily indecomposable continuum with infinite dimensional hyperspace.

Eberhart and Nadler **[EN-1]** have shown that every hereditarily indecomposable continuum has a hyperspace of dimension either two or infinite. Every planar hereditarily indecomposable continuum (as well as every other standard one-dimensional example) has a two-dimensional hyperspace. Every hereditarily indecomposable continuum of dimension at least two has an infinite-dimensional hyperspace.

Lau **[L-1]** has shown that an hereditarily indecomposable continuum X has an infinite-dimensional hyperspace if and only if there exists a monotone map $f\colon X \to Y$ for some Y of dimension greater than 1. In this paper, we prove that every hereditarily indecomposable continuum is the image under an open, monotone map of a one-dimensional hereditarily indecomposable continuum. Thus there exist one-dimensional hereditarily indecomposable continua with infinite-dimensional hyperspaces.

1. Preliminaries. A *continuum* is a nondegenerate compact connected metric space.

An arc is ε-crooked if for each pair of its points p and q there are points r and s between p and q such that r lies between p and s, $\operatorname{dist}(p, s) < \varepsilon$ and $\operatorname{dist}(r, q) < \varepsilon$ **[B-1]**.

We can assume that the target space C_n is embedded in Euclidean space E^{2n+1} (or in the Hilbert Cube Q if n is infinite). Since C_n is hereditarily indecomposable, if U is a cover of C_n (by sets open in E^{2n+1} or Q) and $\varepsilon > 0$, there exists a cover (by sets open in E^{2n+1} or Q) V of C_n, closure refining U, of mesh less than ε, so that every arc in V^* is ε-crooked.

From now on every open set will be open in the appropriate Euclidean space.

If K is a complex, K' will be the one-skeleton of the first barycentric subdivision of K.

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2. Main result.

THEOREM 1. *Let C_n be an n-dimensional hereditarily indecomposable continuum (with n possibly infinite). There exists a one-dimensional hereditarily indecomposable continuum C_1 and a monotone continuous surjection $f\colon C_1 \to C_n$.*

PROOF. We shall construct C_1 inductively.

Let U_1 be a minimal open cover of C_n. U_1^* can be partitioned into disjoint sets $U_1(k)_\alpha = \{x \in U_1^* | \mathrm{st}(x, U_1) = \alpha\}$, where α is a k-element subset of U_1. By slightly modifying some elements of U_1 if necessary, we can guarantee that $\mathrm{cl}(U_1(k)_\alpha) \cap \mathrm{cl}(U_1(j)_\beta) = \varnothing$ unless either $\alpha \subset \beta$ or $\beta \subset \alpha$. Let N_1 be the nerve of U_1.

Let G_1 be a geometric realization of N_1' in E^3 and $\tilde{H}_1$ a regular neighborhood of G_1.

Suppose inductively that we have embedded a regular neighborhood $\tilde{H}_m$ of a piecewise-linear copy of G_m (where G_m is a geometric realization of N_m', with N_m the nerve of an open cover U_m of C_n). We will show how to embed an $\tilde{H}_{m+1}$ in $\mathrm{Int}(\tilde{H}_m)$ satisfying certain conditions which we will point out as we go along.

By subdividing G_m sufficiently and making $\tilde{H}_m$ sufficiently close to the piecewise-linear image $h_m(G_m)$, we can put in splitting disks to give dual cells of diameter less than $1/2^m$ [M-1]. For each vertex v of G_m (before subdivision), $\mathrm{st}(v, G_m')$ consists of a finite number of line segments emanating from vertex v. We can do our subdivision of G_m so that the image under h_m of each of these line segments is covered by the same number, $n_m + 1$, of dual cells.

Choose $1/2^{m+1} > \varepsilon_m > 0$ so small that for each j, k, α, β we have $\mathrm{dist}(U_m(k)_\alpha, U_m(j)_\beta) > \varepsilon_m$ unless either $\alpha \subset \beta$ or $\beta \subset \alpha$. Let U_{m+1} be a minimal open cover of C_n closure refining U_m, of mesh less than $\varepsilon_m/4(n_m + 1)$, so that each arc in U_{m+1}^* is $\varepsilon_m/4(n_m + 1)$-crooked. Form sets $U_{m+1}(i)_\partial$ as above with $\mathrm{cl}(U_{m+1}(i)_\partial) \cap \mathrm{cl}(U_{m+1}(j)_\delta) = \varnothing$ unless either $\partial \subset \delta$ or $\delta \subset \partial$. Let N_{m+1} be the nerve of U_{m+1}, and $\tilde{U}_{m+1} = \{\tilde{U}_{m+1}(i)_\partial \colon \partial \subset U_{m+1}\}$.

Let G_{m+1} be a geometric realization of N_{m+1}' in E^3. Let h_{m+1} be a piecewise-linear embedding of a regular neighborhood H_{m+1} of G_{m+1} into $\mathrm{Int}(\tilde{H}_m)$ satisfying the following.

For each vertex $\tilde{\mu}$ of G_{m+1}, $h_{m+1}(\mathrm{st}(\tilde{\mu}, G_{m+1}'))$ intersects each of the dual cells of $\tilde{H}_m$ covering $h_m(\mathrm{st}(\tilde{\alpha}, G_m))$ if $U_{m+1}(j)_\mu \cap U_m(k)_\alpha \neq \varnothing$, where $\tilde{\mu}$ is the vertex of G_{m+1} corresponding to $U_{m+1}(j)_\mu$ and $\tilde{\alpha}$ is the vertex of $h_m(G_m)$ corresponding to $U_m(k)_\alpha$. In addition, if $n_m \leqslant i \leqslant 2n_m$ and

$$\frac{\varepsilon_m}{4(n_m + 1)} \leqslant \mathrm{dist}(U_{m+1}(j)_\mu, U_m(l)_\beta) < (i + 1)\frac{\varepsilon_m}{4(n_m + 1)}, \tag{A}$$

then $h_{m+1}(\mathrm{st}(\tilde{\mu}, G_{m+1}'))$ intersects each of the dual cells covering the edge between $\tilde{\alpha}$ and $\tilde{\beta}$ which is at least $i - n_m$ dual cells from $\tilde{\beta}$. If $0 \leqslant i < n_m$ and

(A) above, then $h_{m+1}(\mathrm{st}(\tilde{\mu}, G'_{m+1}))$ intersects each of the dual cells covering the edge between $\tilde{\alpha}$ and $\tilde{\beta}$ as well as each dual cell which is within $n_m - i$ dual cells from $\tilde{\beta}$.

Let D_μ be the set of all dual cells included in the above criteria. Then $h_{m+1}(\mathrm{st}(\tilde{\mu}, G'_{m+1})$ intersects the interior of each of these dual cells and intersects no other dual cell not in D_μ. $\mathrm{st}(\tilde{\mu}, G'_{m+1})$ is a finite number of line segments emanating from the vertex $\tilde{\mu}$. The image of each of these line segments intersects the interior of every element of D_μ and is $1/2^m$-crooked. Also, the $h_{m+1}(\mathrm{st}(\tilde{\mu}, G'_{m+1}))$ fit together so that every arc in $h_{m+1}(\mathrm{st}(\mathrm{st}(\tilde{\mu}, G'_{m+1})))$ is $3/2^m$-crooked. Make $h_{m+1}(H_{m+1}) = \tilde{H}_{m+1}$ be a sufficiently small regular neighborhood of $h_{m+1}(G_{m+1})$ that each arc in the part of H_{m+1} corresponding to $\mathrm{st}(\mathrm{st}(\tilde{\mu}, G'_{m+1}))$ is also $3/2^m$-crooked for each μ.

Let $C_1 = \bigcap_{m \geqslant 1} \tilde{H}_m$. C_1 is clearly a one-dimensional continuum. Define f: $C_1 \to C_n$ so that if $x \in \bigcap_{m \geqslant 2} h_m(\mathrm{st}(\mathrm{st}(\tilde{\zeta}_m, G'_m)))^*$ then $f(x) = \bigcap_{m \geqslant 2} \mathrm{st}(U_m(k)_{\zeta m}, \{U_m(j)_\alpha\colon \alpha \subset U_m\})^*$. It is easily checked that f is a continuous monotone surjection. We still need to verify that C_1 is hereditarily indecomposable.

Suppose C is a subcontinuum of C_1 and $C = H \cup K$, where H and K are proper subcontinua of C. Then there exist $h \in H - K$, $k \in K - H$, and $\varepsilon > 0$ so that $\mathrm{dist}(h, K) > \varepsilon$ and $\mathrm{dist}(k, H) > \varepsilon$. Choose m so large that $3/2^m < \varepsilon$.

Let O_H be an open neighborhood of H and O_K an open neighborhood of K so that $\mathrm{dist}(k, O_H) > \varepsilon$, $\mathrm{dist}(h, O_K) > \varepsilon$, and $O_H \cup O_K \subset \tilde{H}_{m+2}$.

Let A be an arc from h to k in $O_H \cup O_K$ so that A is the union of two arcs A_1 and A_2 where $A_1 \subset O_H$, $A_2 \subset O_K$, and $A_1 \cap A_2$ is a single point a.

There are two cases to consider.

Case 1. $f(h) = f(k)$. If there exists μ with $A \subseteq h_{m+1}(H_{m+1}(\mathrm{st}(\mathrm{st}(\tilde{\mu}, G'_{m+1}))))^*$ then A contains points b and c with b between h and c, $\mathrm{dist}(h, c) < 3/2^m$, and $\mathrm{dist}(k, b) < 3/2^m$. Thus the point $a = A_1 \cap A_2$ must be between h and b as well as between c and k, which is a contradiction.

If there is no such μ, then there exists a point d outside of $\mathrm{st}(h, \tilde{H}_{m+1})^*$, and points b and c with b between h and d, c between d and k, $\mathrm{dist}(k, b) < 2/2^m$, and $\mathrm{dist}(h, c) < 2/2^m$. As above, this is a contradiction.

Case 2. $f(h) \neq f(k)$. In this case we can choose m sufficiently large that $\mathrm{st}(\mathrm{st}(f(h), \tilde{U}_m))^* \cap \mathrm{st}(\mathrm{st}(f(k), \tilde{U}_m))^* = \varnothing$. By the crookedness of $\tilde{U}_{m+2}$, there exist points b and c in A with b between h and c, so that $b \in \mathrm{st}(k, \tilde{H}_{m+1})^*$ and $c \in \mathrm{st}(h, \tilde{H}_{m+1})^*$. Then, by our embedding of $\tilde{H}_{m+1}$ in $\tilde{H}_m$, there exist points b' and c' of A with b' between h and c', $\mathrm{dist}(h, c') < 2/2^m$, $\mathrm{dist}(k, b') < 2/2^m$. But this is a contradiction to our choice of $A = A_1 \cup A_2$.

Thus C_1 is hereditarily indecomposable. □

In **[LW-1]** the following theorem was proven.

THEOREM (3.1 PROPOSITION OF [LW-1]). *Let X be a compactum and $\{P_n\}_{n=1}^{\infty}$ be a sequence satisfying*:

(1) *For each n, P_n is a finite collection of nonempty closed subsets of X with $P_n^* = X$, with the elements of P_n having pairwise disjoint interiors, and with* $\mathrm{cl}(\mathrm{int}(p_n)) = p_n$ *for each $p_n \in P_n$.*

(2) *For each $p_{n-1} \in P_{n-1}$,* $\mathrm{st}^4(p_{n-1}, P_n)^* \subseteq \mathrm{st}(p_{n-1}, P_{n-1})^*$.

(3) *There is a positive number L such that for each pair $p_n, p_n' \in P_n$ with $p_n \cap p_n' \neq \varnothing$, $p_n \subseteq N_{L/2^n}(p_n')$.*

(4) *There is positive number K such that for each $p_n \in P_n$ there is a $p_{n-1} \in P_{n-1}$ with $p_n \cap p_{n-1} \neq \varnothing$ and $p_{n-1} \subseteq N_{K/2^n}(p_n)$.*

Let G be defined by $g \in G$ if $g = \bigcap_{n=1}^{\infty} \mathrm{st}(p_n, P_n)^*$ where $\bigcap_{n=1}^{\infty} p_n \neq \varnothing$; G is a continuous decomposition of X. □

Letting P_n be the set of $\mathrm{st}(\tilde{\mu}, \tilde{H}_n) \cap C_1$, it is easily checked that the above conditions are satisfied (with $X = C_1$). Since the g's of the above theorem are precisely the point inverses under the map f of Theorem 1, they form a continuous decomposition of C_1 and f is open, monotone.

COROLLARY. *The hyperspace $C(C_1)$ of subcontinua of C_1 contains a copy of C_n, so $C(C_1)$ has dimension at least n.* □

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