PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 2, July 1979

SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A SHORT PROOF OF A CARDINAL INEQUALITY INVOLVING HOMOGENEOUS SPACES

RYSZARD FRANKIEWICZ

ABSTRACT. We give a short proof of van Douwen's theorem that $|X| \leqslant 2^{\pi(X)}$ holds for homogeneous Hausdorff spaces.

Let X be a Hausdorff space. A *π-basis* for X is a collection $\mathfrak{U}$ of nonempty open subsets of X such that each nonempty open subset of X contains a member of $\mathfrak{U}$. The *π-weight* $\pi(X)$ of X is the least cardinality of a π-basis for X. Finally, $H(X)$ denotes the group of all autohomeomorphisms of X. In this note we show that the cardinality of $H(X)$ cannot exceed $2^{\pi(X)}$. This slightly generalizes a result due to van Douwen [vD], while our proof is much easier. The author is grateful to Jan van Mill and Charles F. Mills for assistance in preparing this note.

THEOREM. *$|H(X)| \leqslant 2^{\pi(X)}$ whenever X is a Hausdorff space.*

PROOF. Let $\mathfrak{U}$ be a π-basis for X of cardinality $\pi(X)$. For each $\varphi \in H(X)$ define $\varphi^{\#}\colon \mathfrak{U} \to \mathcal{P}(\mathfrak{U})$ by

$$\varphi^{\#}(U) = \{ V \in \mathfrak{U} \mid V \subset \varphi[U] \}.$$

Clearly $\varphi \neq \psi$ implies that $\varphi^{\#} \neq \psi^{\#}$. Hence

$$|H(X)| \leqslant |\mathcal{P}(\mathfrak{U})^{\mathfrak{U}}| = (2^{\pi(X)})^{\pi(X)} = 2^{\pi(X)}. \qquad \square$$

COROLLARY. *If X is a homogeneous Hausdorff space then $|X| \leqslant 2^{\pi(X)}$.*

REMARKS. One useful application of the corollary is in proving that certain spaces are not homogeneous. For example, if X is nowhere locally compact and $\pi(X) = \omega$, then $\pi(\beta X - X) = \omega$; then if X is nonpseudocompact, $|\beta X - X| = 2^{2^{\omega}} > 2^{\omega}$ so, by the corollary, $\beta X - X$ is not homogeneous. Among the familiar examples of such spaces are the rationals, the irrationals, and the Sorgenfrey line. These observations are due to van Douwen [vD].

Received by the editors November 6, 1978 and, in revised form, December 5, 1978.
AMS (MOS) subject classifications (1970). Primary 54H99.
Key words and phrases. π-weight, homogeneous.

0002-9939/79/0000-0338/$01.50

REFERENCES

[vD] E. K. van Douwen, *Nonhomogeneity of products of preimages and π-weight*, Proc. Amer. Math. Soc. 69 (1978), 183–192.

[J] I. Juhász, *Cardinal functions in topology*, Mathematical Centre Tract 34, Math. Centrum, Amsterdam, 1971.

INSTYTUT MATEMATYKI, UNIWERSYTET SLASKI, UL. BANKOWA 14, 40007 KATOWICE, POLAND.