PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 1, August 1979

THE ESSENTIAL CLOSURE OF $C(X)$

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ABSTRACT. Each archimedean l-group admits a unique essential closure, which is the l-group of continuous almost finite real-valued functions on some Stonean space; thus the l-group $C(X)$ of real-valued continuous functions on a topological space X admits such an essential closure. In this note we will construct a natural embedding of $C(X)$ into its essential closure, making explicit the topological relationship between X and the appropriate Stonean space.

1. Preliminaries. Throughout this paper G will denote an archimedean lattice-ordered group (l-group). A general reference on l-groups is **[2]**. If G is an l-subgroup of an l-group H, then G is *large* in H (or H is an *essential extension* of G) if $C \cap G \neq 0$, for each convex l-subgroup C of H. An archimedean l-group is *essentially closed* if it admits no proper archimedean essential extensions; H is an *essential closure* of G if H is essentially closed and an essential extension of G. That each archimedean l-group admits a unique essential closure is due to Conrad **[5]**; this closure is of the form $D(Y)$, where Y is a Stonean space (that is, compact Hausdorff and extremally disconnected). Here $D(Y) = \{f\colon Y \to \mathbf{R}^*\colon f^{-1}(\mathbf{R})$ is dense in Y and f is continuous$\}$, where $\mathbf{R}^* = \mathbf{R} \cup \{\pm\infty\}$ is the two-point compactification of the real numbers $\mathbf{R}$. Bernau **[1]** first showed that any archimedean l-group could be embedded into such an l-group.

For $K \subseteq G$, let

$$K' = \{g \in G\colon |g| \wedge |x| = 0, \text{ for } x \in K\}.$$

Then

$$P(G) = \{C''\colon C \text{ is a convex } l\text{-subgroup of } G\}$$

is the set of *polars* of G; $P(G)$ is a subset of the set of convex l-subgroups of G, and is a complete Boolean algebra, with set-theoretic intersection for meet, and $'$ for complementation. If H is archimedean and $G \subseteq H$, then G is large in H precisely when the natural intersection map $C \to C \cap G$ is a Boolean algebra isomorphism between $P(H)$ and $P(G)$ **[4]**.

Given a Tychonoff space X, let ΘX be the set of all regularly open ultrafilters on X. If U is a regularly open subset of X, let $\mathcal{O}(U) = \{p \in \Theta X\colon U \in p\}$. The set of all such $\mathcal{O}(U)$ forms a base for the Stone topology on ΘX, which makes ΘX a Stonean space **[6]**. Furthermore, $\mathcal{O}: \mathfrak{R}(X) \to \mathfrak{R}(\Theta X)$

Received by the editors March 14, 1979.
AMS (MOS) subject classifications (1970). Primary 06A60; Secondary 54C40.

0002-9939/79/0000-0351/$01.75

is a Boolean algebra isomorphism between the regularly open subsets of X and of ΘX [**3**, p. 40]. Now let $\omega X = \{p \in \Theta X\colon p \text{ is fixed}\}$, which is called the absolute of X. Then ωX is a dense subset of ΘX [**6**], and so $\beta(\omega X) = \Theta X$. Define $\pi\colon \omega X \to X$ by letting $\pi(p)$ be the (unique) point to which p converges. Then π is a continuous function [**6**].

2. The embedding. We will need the following result which characterizes polars topologically:

PROPOSITION. *Let X be a Tychonoff space. Let $G = C(X)$ (or $D(X)$, where X is Stonean). Then the map $\tau(X)\colon P(G) \to \mathfrak{R}(X)$ defined by*

$$\tau(X)(C) = \text{Interior}(\text{Closure}\{x \in X\colon f(x) \neq 0, \textit{some } f \in C\})$$

is a Boolean algebra isomorphism.

This proposition is well known and its proof will be omitted.

THEOREM. *If X is a Tychonoff space, then $\alpha\colon C(X) \to D(\Theta X)$, defined so that the following diagram commutes for all $f \in C(X)$, is a large l-embedding of $C(X)$ into its essential closure*:

$$\begin{tikzcd}
\beta(\omega X) = \Theta X \arrow[drrr, "\alpha(f) = \beta(\iota \circ f \circ \pi)"] & & & \\
\omega X \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\pi"] & X \arrow[r, "f"] & \mathbf{R} \arrow[r, tail, "\iota"] & \mathbf{R}^*
\end{tikzcd}$$

PROOF. Since $\alpha(f)^{-1}(\mathbf{R})$ is an open subset of ΘX which contains ωX and is consequently dense, $\alpha(f)$ is an element of $D(\Theta X)$. Given $f, g \in C(X)$, $\alpha(f) + \alpha(g)$ is defined as the unique extension of $\alpha(f)|U + \alpha(g)|U$ to ΘX, where $U = \alpha(f)^{-1}(\mathbf{R}) \cap \alpha(f)^{-1}(\mathbf{R})$, an open dense set. (This unique extension exists because X is Stonean.) But $\alpha(f) + \alpha(g)$ then agrees with $\alpha(f + g)$ on ωX, a dense subset of ΘX, and so $\alpha(f) + \alpha(g) = \alpha(f + g)$. A similar argument shows that α preserves the lattice operations. Since α is clearly monic, it remains to show that $C(X)$ (identified now with $\alpha(C(X))$) is large in $D(\Theta X)$. But

$$\tau(X)^{-1}\mathcal{O}^{-1}\tau(\Theta X)\colon P(D(\Theta X)) \to P(C(X))$$

is a Boolean algebra isomorphism, and

$$\tau(X)^{-1}\mathcal{O}^{-1}\tau(\Theta X)(C) = C \cap C(X).$$

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