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WEAK q-RINGS WITH ZERO SINGULAR IDEAL

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ABSTRACT. A ring R is called a (right) wq-ring if every right ideal not isomorphic to R_R is quasi-injective. The main result proved is the following: Let R be a ring with zero singular ideal, then R is a wq-ring if and only if either R is a q-ring, or $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ for some division ring D, or R is such that every right ideal not isomorphic to R_R is completely reducible.

Throughout this paper, the rings considered are with unity and every module is a unital right module. A ring R is called a (right) q-ring if every right ideal of R is quasi-injective [5]. Such rings and their dual concept have been studied by many authors. Recently Byrd [1] determined the structure of q-rings without imposing any finiteness conditions. In [8], the present authors initiated the study of those rings R whose right ideals not isomorphic to R_R are quasi-injective; such rings are called weak q-rings (in short wq-rings). The structure of wq-rings under some finiteness conditions was determined in [8]. In this paper we study wq-rings with zero right singular ideal, which need not satisfy any finiteness conditions. A characterization of such rings is given in Theorem (2.9). The structures of the right socle and the Jacobson radical of these rings are also determined.

1. Preliminaries. For definition and some properties of quasi-injective modules, we refer the reader to Johnson and Wong [6] (see also Faith [2]). For any module M_R, the smallest cardinal α such that any direct sum in M_R has at most α components is called the dimension of M (denoted by $d(M)$). A submodule N of a module M is called a complement submodule if N is a complement of some submodule K of M. The following two results are due to Miyashita [7]:

(1.1) THEOREM. *Any complement submodule of a quasi-injective module M is a summand of M.*

(1.2) THEOREM. *A finite dimensional quasi-injective module is a direct sum of uniform modules.*

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Ivanov [**4**] proved the following:

(1.3) LEMMA. *Let A and B be right ideals of a q-ring R. If $A \cap B = 0$ then for any R-homomorphism f: $A \to B$, $f(A)$ is semisimple (i.e., completely reducible).*

On similar lines, the following can be proved:

(1.4) LEMMA. *Let A and B be any modules such that for every essential submodule C of B, $A \oplus C$ is quasi-injective. Then for any homomorphism f: $A \to B$, $f(A)$ is completely reducible.*

The following is well known:

(1.5) LEMMA. *Let A and B be any modules. If $A \oplus B$ is quasi-injective, then any monomorphism from A to B splits.*

For any module M_R, $Z(M)$ and $\hat{M}$ will denote the singular submodule and the injective hull of M, respectively. It is well known that if a ring R has $Z(R) = 0$, then $\hat{R}$ is a right self-injective ring of which R is a subring (see [**2**]).

For any right ideal A of a ring R with $Z(R) = 0$, let

$$A^* = \{x \in R \colon \hat{R}x \subset A\}.$$

Then A^* is a right ideal of R contained in A. It is proved in [**8**, Lemma (1.1)] that A^* is a left ideal in $\hat{R}$ and is a quasi-injective R-module; if in addition A is essential in R_R, then $A = A^*$ if and only if A is quasi-injective.

2. Rings with zero singular ideal. We start by the following general result.

(2.1) LEMMA. *Let A be a quasi-injective right ideal in a wq-ring R. If A contains a right regular element, then R is a q-ring.*

PROOF. Let $a \in A$ be a right regular element. Then $R \simeq aR \subset A$. This implies that A is injective and $\hat{R}$ is embedded in A. Let $R = B \oplus C$ where $B \simeq \hat{R}$. Then $B = B_1 \oplus B_2$ where $B_1 \simeq B$ and $B_2 \simeq \hat{C}$. If $B_2 \oplus C \not\simeq R$, then $B_2 \oplus C$ is quasi-injective. Since C embeds in B_2, then Lemma (1.5) gives that C is injective. Hence R_R is injective and R is a q-ring. On the other hand, $B_2 \oplus C \simeq R$ implies that $R \simeq \hat{R} \oplus R$. This in turn implies that R contains an infinite direct sum $\Sigma \oplus R_i$ of copies of R. Since $\Sigma \oplus R_i \not\simeq R$, we get $\Sigma \oplus R_i$ is quasi-injective. Hence R_R is injective. This completes the proof.

The following is proved in [**8**, Theorem (2.7)].

(2.2) THEOREM. *Let R be a ring such that $Z(R) = 0$. If R is a wq-ring, then either R is a right PID, or R is a strongly regular right self-injective ring (hence a q-ring) or R has nonzero right socle.*

Since right PID's and q-rings are trivial cases of wq-rings, we will be mainly interested in those wq-rings which are not of these two types. As such, in all the lemmas to follow, R will be a wq-ring with $Z(R) = 0$ which is neither a right PID nor a q-ring. S will denote the right socle of R. We note that R has no nontrivial central idempotents [**8**, Lemma (1.5)].

(2.3) LEMMA. *S is essential in R_R.*

PROOF. By Theorem (2.2), $S \neq 0$. Suppose S is not essential in R_R. Let C be a complement of S. We claim that $S \oplus C \not\simeq R$. Suppose not. Then $S = gR$ for some nontrivial idempotent g. This implies that every minimal right ideal of R is projective, and hence

$$gR(1 - g) = 0 = (1 - g)Rg.$$

Thus g is a central idempotent, a contradiction. Therefore $S \oplus C \not\simeq R$. Hence $S \oplus C = (S \oplus C)^*$, a left ideal in $\hat{R}$. Let aR be a minimal right ideal of R. There exist $x \in \hat{R}$ such that $axa = a$. Then $f = xa$ is an idempotent in R and $aR \simeq fR$. This proves that every minimal right ideal of R is projective. We proceed to show that C is a left ideal in $\hat{R}$. On the contrary, let $y \in \hat{R}$ be such that $yC \not\subset C$. However $yC \subset S \oplus C$. This defines a nonzero homomorphism η: $C \to S$. Since $\operatorname{Im} \eta$ is projective and completely reducible, we get $S \cap C \neq 0$. This is a contradiction. Hence $\hat{R}C \subset C$. As $\hat{R}$ is a regular ring, C contains a nontrivial idempotent e. Then $C = eR \oplus (1 - e)C$. Since $\operatorname{Soc}(C_R) = 0$, Lemma (1.4) gives

$$\operatorname{Hom}(eR, (1 - e)C) = 0 = \operatorname{Hom}((1 - e)C, eR).$$

Consequently,

$$(1 - e)Ce = 0 = eR(1 - e)C.$$

Now, $eR(S \oplus (1 - e)C) = eRS + eR(1 - e)C = 0$. So that

$$S \oplus (1 - e)C \subset (1 - e)R.$$

Since $S \oplus C \subset ' R$, we get $S \oplus (1 - e)C \subset '(1 - e)R$. Then $Z(R) = 0$ implies $eR(1 - e)R = 0$. Also $(1 - e)Re = (1 - e)Ce = 0$. Hence e is a central idempotent, a contradiction. This completes the proof.

(2.4) COROLLARY. *Let R be a wq-ring with $Z(R) = 0$ which is neither a right PID nor a q-ring. Then any minimal right ideal of R is projective and every homogeneous component of* $\operatorname{Soc}(R_R)$ *contains nonzero idempotents.*

(2.5) PROPOSITION. *If R is a wq-ring with $Z(R) = 0$, then R/R^* is a right PID.*

PROOF. The result is obvious for a right PID or a q-ring. So assume that R is not one of these rings. By Lemma (2.3), $S = \operatorname{Soc}(R_R) \subset ' R_R$. Then $S = S^* \subset R^*$, and hence $R^* \subset ' R_R$. Let A/R^* be a nonzero right ideal of R^*. Then $A \supsetneqq R^*$ implies that $A \simeq R$. So that $A = aR$ for some right regular element a of R. Since $\hat{R}$ is a regular ring, a has a left inverse in $\hat{R}$. Hence $aR^* = aR \cap R^*$. Now

$$A/R^* = (aR + R^*)/R^* \simeq aR/(aR \cap R^*) = aR/aR^* \simeq R/R^*.$$

This proves that every nonzero right ideal of R/R^* is isomorphic to R/R^*. Hence R/R^* is a right PID.

(2.6) PROPOSITION. *Let R be a wq-ring which is neither a right PID nor a q-ring. If $d(R) > 2$, then $d(R) = \infty$; in fact, no homogeneous component of* $\mathrm{Soc}(R_R)$ *is finitely generated.*

PROOF. Let H be a homogeneous component of S. By Corollary (2.4), there exists an idempotent $e \in H$ such that eR is a minimal right ideal. Suppose that $d(H) < \infty$. If $(1-e)R \simeq R$, then $(1-e)H = (1-e)R \cap H \simeq H$. This contradicts $d(H) < \infty$. Hence $(1-e)R \not\simeq R$, and $(1-e)R$ is quasi-injective. Note that $(1-e)H \neq 0$, since otherwise $H = eR$ and e becomes a central idempotent. Let B be the maximal essential extension of $(1-e)H$ in $(1-e)R$. Then by Theorem (1.1), $(1-e)R = B \oplus C$ for some right ideal C. Then B is quasi-injective and $d(B) < \infty$. By Theorem (1.2),

$$B = U_1 \oplus U_2 \oplus \cdots \oplus U_t$$

for finitely many uniform submodules U_i. Since $d(R) > 2$, $eR \oplus U_i \not\simeq R$, and so $eR \oplus U_i$ is quasi-injective. Then as eR is embeddable in U_i, $eR \simeq U_i$ by Lemma (1.5). This gives $B = (1-e)H$, and so H is a summand of R. This is again a contradiction. Hence the result follows.

(2.7) LEMMA. *If $d(R) > 2$ and if B is any right ideal of R such that $B \cap S$ is finitely generated, then $B \subset S$.*

PROOF. Since $S \subset' R_R$ by Lemma (2.3), $B \cap S \subset' B$. Then $B \cap S$ is finitely generated implies $d(B) < \infty$. However $d(R) = \infty$ by the above proposition. Consequently B is quasi-injective, and by Theorem (1.2), $B = \sum_{i=1}^{t} \oplus U_i$, with each U_i uniform right ideal. Since each homogeneous component of S has infinite dimension by Proposition (2.6), we can find minimal right ideals A_i such that $A_i \cap U_i = 0$ and $A_i \simeq \mathrm{Soc}(U_i)$. Then $A_i \simeq U_i$ by the quasi-injectivity of $A_i \oplus U_i$. This proves that $B \subset S$.

(2.8) LEMMA. *If $d(R) > 2$, then any right ideal A of R not isomorphic to R, is completely reducible.*

PROOF. Let $a \in A$. By Lemma (2.1), $aR \not\simeq R$ and consequently aR is quasi-injective. We claim that every homogeneous component of $\mathrm{Soc}(aR \cap S)$ is finitely generated. Suppose not. Then we get a direct sum

$$\sum_{i=1}^{\infty} \oplus A_i \oplus \sum_{i=1}^{\infty} \oplus B_i$$

of mutually isomorphic minimal right ideals A_i and B_i contained in aR. Let D be a complement submodule of $\Sigma \oplus A_i$ in aR containing $\Sigma \oplus B_i$. Then by Theorem (1.1), D is a summand of aR. Now $\Sigma \oplus A_i \oplus D$ is quasi-injective and $\Sigma \oplus A_i$ is embeddable in D; so by Lemma (1.5), $\Sigma \oplus A_i$ is isomorphic to a summand of D. Consequently $\Sigma \oplus A_i$ is finitely generated; a contradiction. Hence every homogeneous component of $\mathrm{Soc}(aR \cap S)$ is finitely generated.

Now assume that aR is not completely reducible. Then by Lemma (2.7),

$d(aR \cap S) = \infty$. Hence $\mathrm{Soc}(aR \cap S)$ contains infinitely many homogeneous components. Since no homogeneous component of S is finitely generated by Proposition (2.6), we can find an infinite direct sum $\Sigma \oplus C_i$ of minimal right ideals such that $(\Sigma \oplus C_i) \cap aR = 0$ and $\Sigma \oplus C_i$ embeds in aR. However, this is a contradiction, since $aR \oplus \Sigma \oplus C_i$ is quasi-injective. Therefore aR is completely reducible. This completes the proof.

We now prove the main theorem.

(2.9) THEOREM. *Let R be a ring with $Z(R) = 0$. Then R is a wq-ring if and only if*:

(1) *R is a q-ring, or*

(2) *$R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ for some division ring D, or*

(3) *Every right ideal of R, not isomorphic to R is completely reducible.*

PROOF. Let R be a wq-ring. Suppose R is not a q-ring. If $d(R)$ is finite, then by [**8**, Theorem (2.4)], R is of type (2), or R is a right PID and as such is of type (3). If $d(R)$ is infinite, then by the above Lemma, R is of type (3). This proves the necessity. The sufficiency is obvious.

The following is a consequence of the above theorem and Corollary (2.4).

(2.10) THEOREM. *If R is a wq-ring with $Z(R) = 0$ and if R is not a q-ring, then R is right hereditary.*

The next proposition gives some information about the socle and the Jacobson radical of wq-ring R with $Z(R) = 0$.

(2.11) PROPOSITION. *Let R be a wq-ring with $Z(R) = 0$ and let J be its Jacobson radical. Then $J^2 = 0$. Further if R is not a q-ring and if $d(R) > 2$, then every homogeneous component of* $\mathrm{Soc}(R_R)$ *contains an infinite number of orthogonal idempotents.*

PROOF. It is obvious from [**8**, Theorem (2.4)] that if R is a q-ring or $d(R) \leqslant 2$, then $J^2 = 0$. So we assume that R is not a q-ring and $d(R) > 2$. Let $A = J \cap S$, then $A^2 = 0$. Write $S = A \oplus B$ for some right ideal B of R. Then B is a sum of minimal right ideals each of which is generated by an idempotent. Further by Corollary (2.4), every homogeneous component of S has nonzero intersection with B. Let H be a homogeneous component of B. If H is finitely generated, then $H = eR$ for some idempoent $e \in R$. Then $(1 - e)R$ is not completely reducible. It follows by Theorem (2.9) that $(1 - e)R \simeq R$. Consequently $(1 - e)R = C \oplus D$ with $C \simeq H = eR$ and $D \simeq (1 - e)R$. Then

$$R = eR \oplus C \oplus D.$$

Since $SJ = 0, J = DJ$ and hence $(eR \oplus C) \cap J = 0$. Now

$$S = eR \oplus C \oplus (D \cap S) = B \oplus A.$$

This defines a monomorphism $\theta\colon eR \oplus C \to B$. It is clear that $\mathrm{Im}\,\theta \subset H$ and $d(eR \oplus C) = 2d(H)$. This is a contradiction. Hence every homogeneous component of B contains an infinite direct sum. Then $J \oplus B \not\simeq R$, and by Lemma (2.8), $J \subset S$. Hence $J = A$ and $J^2 = 0$. This completes the proof.

Harada [3] proved that if for two modules M_1 and M_2, $M_1 \oplus M_2$ is quasi-injective, then $M_1 \simeq M_2$ if and only if $\hat{M}_1 \simeq \hat{M}_2$. Using this and Theorem (2.9), one can easily prove the following:

(2.12) PROPOSITION. *Let R be a wq-ring with $Z(R) = 0$. Then $\hat{R}$ is a prime ring if and only if socle (R_R) is homogeneous.*

We end this paper with the following remarks.

(1) We are not aware of an example of a ring R with $Z(R) = 0$, in which every right ideal not isomorphic to R is completely reducible, but the ring itself is neither a right PID nor semisimple artinian.

(2) Since a semiprime *wq*-ring has zero right singular ideal [**8**, Proposition (1.8)], the results on semiprime, and in particular prime, *wq*-rings proved in [**8**], are immediate consequences of results established above.

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