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GROUP ACTIONS ON Q-F-RINGS

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ABSTRACT. Let B be a ring, G a finite group of automorphisms acting on B and B^G the fixed subring of B. We give an example of a B which is quasi-Frobenius (Q-F) such that B^G is not quasi-Frobenius.

S. Jøndrup [3] claims that if card G is a unit in B and if B is self-injective then B^G is self-injective. J. Fisher and J. Osterburg [2] use this assertion to prove that if B is quasi-Frobenius then B^G is quasi-Frobenius. However we show that this result fails.

Suppose that A is a commutative local artinian ring with Jacobson radical R and call E the injective hull of the simple A-module $S = A/R$. E is finitely generated and faithful [5, théorème 2, p. 97 and corollaire 6, p. 99]. B denotes the ring constructed on the abelian group $A \times E$ with the multiplication

$$(a, e)(a', e') = (aa', ae' + a'e).$$

LEMMA. *B is a commutative local quasi-Frobenius ring.*

It is clear that B is commutative local artinian with radical $R \times E$ and that $S' = O \times S$ is a simple ideal of B essential in the ideal $O \times E$. $O \times E$ is essential in B, since for each $a \in A - \{o\}$ there exists $e' \in E$ such that $(o, e')(a, e) = (o, ae')$ is nonzero, because E is faithful. Thus S' is essential in B.

Then for each $f \in \mathrm{Hom}_B(S', B) - \{o\}$, $f(S')$ and S' are equal so that $\mathrm{Hom}_B(S', B)$ is isomorphic to S'. By [1, Theorem 58.6, p. 396], B is quasi-Frobenius. $\square$

PROPOSITION. (1) *There exists a quasi-Frobenius ring B and a finite group G of automorphisms of B with* card G *invertible in B such that B^G is not quasi-Frobenius.*

(2) *It gives also an example of a quasi-Frobenius ring C with an idempotent e such that eCe is not quasi-Frobenius.*

(1) Consider a field K of characteristic different from 2 and define on K^3 a ring A by the multiplication

$$(a, b, c)(a', b', c') = (aa', ab' + a'b, ac' + a'c).$$

A is a commutative local artinian ring whose socle has length 2. Thus A is not quasi-Frobenius.

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Define B as before. $(a, e) \mapsto (a, -e)$ is an automorphism of B of order 2 with fixed subring A.

(2) As in [2] consider the "twisted" group ring C defined on the set of all formal sums $\sum_{g \in G} b_g g$ with $b_g \in B$, by the multiplication $g \cdot r = r^g g$; $e = (\sum_{g \in G} g)/\text{card } G$ is an idempotent of C and eCe is isomorphic to $B^G = A$ [2]. But it is easy to adapt the classical demonstration of injectivity of group rings [5, pp. 103–104] to prove that C is quasi-Frobenius. $\square$

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