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# A REMARK ON SCHUR INDICES OF $p$-GROUPS

TOSHIHIKO YAMADA

ABSTRACT. By making use of Hasse's sum theorem, a simple proof of the following theorem on Schur indices of $p$-groups is given.

THEOREM (ROQUETTE [3] AND SOLOMON [4]). *Let $p$ be a prime number, $G$ a $p$-group, and $\chi$ an irreducible complex character of $G$. Let $m_Q(\chi)$ denote the Schur index of $\chi$ over the rational field $Q$. Then, $m_Q(\chi) = 1$ for $p \neq 2$, and $m_Q(\chi) = 1$ or $2$ for $p = 2$.*

PROOF. Let $A$ be the simple component of the group algebra $Q[G]$, which corresponds to $\chi$. The center of $A$ is $Q(\chi)$, the extension field of $Q$ generated by the elements $\{\chi(g);\ g \in G\}$. Put $k = Q(\chi)$. Let $q$ be a rational prime (possibly the infinite prime $\infty$) and $\mathfrak{q}$ a prime of $k$, lying above $q$. Let $\operatorname{inv}_{\mathfrak{q}}(\chi)$ denote the Hasse invariant of $A$ at $\mathfrak{q}$. It is well known that if $q \neq p, \infty$, then $\operatorname{inv}_{\mathfrak{q}}(\chi) \equiv 0 \pmod 1$, i.e., the Schur index $m_{Q_q}(\chi) = 1$, $Q_q$ being the $q$-adic numbers. (A result which may be established by means of modular representation theory for the prime $q \nmid |G|$.)

Let $|G| = p^n$ and $\zeta$ a primitive $p^n$th root of unity. Then $k = Q(\chi) \subset Q(\zeta)$. Hence there is only one prime $\mathfrak{p}$ of $k$ lying above $p$ (cf. Theorem 1 of [2, p. 73]). Let $\mathfrak{p}_{\infty,1}, \ldots, \mathfrak{p}_{\infty,s}$ be the infinite primes of $k$. Hasse's sum theorem (Satz 9, p. 119 of [1]) now yields that

$$\operatorname{inv}_{\mathfrak{p}}(\chi) + \sum_{i=1}^{s} \operatorname{inv}_{\mathfrak{p}_{\infty,i}}(\chi) \equiv 0 \pmod 1.$$

Since $\operatorname{inv}_{\mathfrak{p}_{\infty,i}}(\chi) \equiv 0$ or $\frac{1}{2}$, it follows that $\sum_{i=1}^{s} \operatorname{inv}_{\mathfrak{p}_{\infty,i}}(\chi) \equiv 0$ or $\frac{1}{2} \pmod 1$, and consequently $\operatorname{inv}_{\mathfrak{p}}(\chi) \equiv 0$ or $\frac{1}{2} \pmod 1$. This implies that $m_Q(\chi) = 1$ or $2$. Since $m_Q(\chi) | p^n$, then $m_Q(\chi) = 1$ for $p \neq 2$.

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, FUKAZAWA, SETAGAYA, TOKYO 158, JAPAN

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