PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 1, August 1979

AN EXTREME POINT IN $H^\infty(U^2)$

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ABSTRACT. In this paper an example of a function $f \in H^\infty(U^2)$ with $\|f\|_\infty = 1$ and

$$\int_{T^2} \log(1 - |f^*(z)|)\,dm > -\infty, \qquad z \in T^2,$$

yet f is an extreme point in the unit ball of H^∞, is given. For functions $f \in H^\infty(U^1)$, that

$$\int_T \log(1 - |f^*(z)|)\,dm = -\infty, \qquad z \in T,$$

is both *necessary and sufficient* for f to be an extreme point in H^∞.

If $\|f\|_\infty = 1$, then a necessary and sufficient condition for $f \in H^\infty(U^1)$ to be an extreme point in the unit ball of H^∞ is that

$$\int_T \log(1 - |f^*(e^{i\theta})|)\,d\theta = -\infty$$

[**1**, pp. 138–139].

If $\|f\|_\infty = 1$, and $f \in H^\infty(U^N)$, the integral condition

$$\int_{T^N} \log(1 - |f^*(z)|)\,dm = -\infty, \qquad z \in T^N,$$

where dm is normalized Haar measure on T^N, still implies that f is an extreme point in the unit ball of H^∞. The proof is the same as in one variable [**1**, pp. 138–139].

The purpose of this paper is to give an example of a function which is extreme in $H^\infty(U^2)$, yet the corresponding integral over T^2 is finite.

EXAMPLE. There exists a function $f \in H^\infty(U^2)$, $\|f\|_\infty = 1$, with

$$\int_{T^2} \log(1 - |f^*(z)|)\,dm > -\infty, \qquad z \in T^2,$$

yet f is extreme in H^∞.

PROOF. First, let S be a compact subset of the real line with positive Lebesgue measure. Choose $\alpha > 0$ and irrational, and consider the function

$$\Phi(\lambda) = (e^{i\alpha\lambda}, e^{i\lambda}), \qquad \operatorname{Im} \lambda \geqslant 0.$$

Φ maps the open half-plane P into U^2, and Φ is continuous on the closure $\bar{P}$ of P.

Received by the editors September 30, 1978.

AMS (MOS) subject classifications (1970). Primary 32A10, 46J15.

Key words and phrases. Extreme point, polydisc algebra, sup norm, integral log condition finite.

Let $E = \Phi(S)$; if $f \in H(U^2)$ and the nontangential limits of f on $E = 0$, then $f \equiv 0$.

Fix r, $0 < r < 1$. Then $\Phi(P)$ contains all points of the form $(r^\alpha e^{i\alpha x}, re^{ix})$, $-\infty < x < \infty$. These points form a dense subset of a certain torus (center at (0, 0), radii r^α and r), since f is continuous on this torus, f vanishes at every point of the torus; this forces f to be identically 0 on U^2.

With E as above, choose $\psi \in C(T^2)$, $\psi = 0$ on E, $\psi < 0$ on the rest of T^2, and

$$\int_{T^2} \log(1 - \exp 2\psi)dm > -\infty.$$

Since ψ is continuous, there is a constant c, such that $\psi + c > 0$ on T^2. By [2, Theorem 3] there exists a positive singular measure σ such that $\tilde{u} = P[\psi + c - d\sigma]$ is the real part of a holomorphic function (thus $\in RP$) in U^2. But $P[\psi + c - d\sigma] = c + P[\psi - d\sigma]$; hence $u = P[\psi - d\sigma] \in RP$. Let $f = \exp(u + iv)$ where $u + iv \in H(U^2)$. Then

$$f \in H^\infty(U^2), \tag{1}$$

$$|f| \leqslant 1, \tag{2}$$

$$|f^*| = \exp u^* = \exp \psi \quad \text{a.e. on } T^2 \tag{3}$$

[(1) $|f| = \exp u = \exp P[\psi - d\sigma] \leqslant \exp P[\psi]$ in U^2. (2) follows since $\psi < 0$ on T^2. (3) follows since σ is singular and $P[d\sigma]$ has radial limits 0 a.e.], and

$$\int_{T^2} \log(1 - |f^*(z)|)dm > -\infty, \qquad z \in T^2,$$

since

$$\begin{aligned}\int \log(1 - \exp 2\psi)dm &= \int \log(1 - \exp \psi)dm + \int \log(1 + \exp \psi)dm \\ &= \int \log(1 - |f^*|)dm + \int \log(1 + \exp \psi)dm > -\infty.\end{aligned}$$

Suppose $g \in H^\infty(U^2)$ and $\|f \pm g\|_\infty \leqslant 1$; then $|g|^2 \leqslant 1 - |f|^2$ and hence $|g^*|^2 \leqslant 1 - |f^*|^2 = 1 - \exp 2\psi$ a.e. Hence

$$|g| = |P[g^*]| \leqslant P[|g^*|] \leqslant P[(1 - \exp 2\psi)^{1/2}].$$

g has nontangential limits 0 on E, and hence $g \equiv 0$, and f is consequently extreme.

Bibliography

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