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THE MILNOR SIGNATURES OF COMPOUND KNOTS

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ABSTRACT. The Milnor signatures of a classical knot are related to those of its companions.

1. We shall work throughout in the smooth or piecewise-linear category. A *knot* k is a circle S^1 embedded in the 3-sphere S^3. A regular neighbourhood V of k is a solid torus. A *longitude* of ∂V is a circle embedded in ∂V which is homologous to k in V, and null-homologous in the closed complement of V. We assume that all knots and longitudes are oriented.

Let T be a solid torus unknotted in S^3 and containing a knot l^*, and let f be a faithful map from T onto V (that is, a homeomorphism which takes a longitude of ∂T onto a longitude of ∂V). If l^* represents $n \in \mathbf{Z} = H_1(T)$, then $l = f(l^*)$ is homologous to nk in V. Proofs will be presented as though n were positive, but with trivial adjustments in notation they are valid for all n.

Let $\Delta_l(t)$ be the Alexander polynomial of the knot l, and $\Delta_k(t)$, $\Delta_{l^*}(t)$ those of k, l^* respectively. It is a result of Seifert [S] that $\Delta_l(t) = \Delta_k(t^n) \cdot \Delta_{l^*}(t)$.

If $p(t)$ is a symmetric, quadratic factor of $\Delta_k(t)$, irreducible over the real numbers, then we can write $p(t)$ in the form $t^{-1} - 2\cos\theta + t$, $0 < \theta < \pi$. Milnor [$\mathbf{M_1}$] has defined a signature $\sigma_\theta(k)$ which is an invariant of k. Let $p(t^n) = p_1(t) \cdots p_n(t)$, where each $p_r(t)$ is symmetric, quadratic and irreducible over the real numbers, and let $\exp(i\theta_r)$ be the root of $p_r(t)$ which is also an nth root of $\exp(i\theta)$, where $0 < |\theta_r| < \pi$.

THEOREM. $\sigma_{|\theta_r|}(l) = \sigma_{|\theta_r|}(l^*) + \sigma_\theta(k)\operatorname{sign}(n \sin\theta_r)$, *where* $\sigma_{|\theta_r|}(l^*) = 0$ *if* $\exp(i\theta_r)$ *is not a root of* $\Delta_{l^*}(t)$.

If $\exp(i\varphi)$ *is a root of* $\Delta_{l^*}(t)$ *but not of* $\Delta_k(t^n)$, *then* $\sigma_\varphi(l) = \sigma_\varphi(l^*)$.

COROLLARY.

$$\begin{aligned} \sigma(l) &= \sigma(l^*) \quad \textit{if } n \textit{ is even}, \\ &= \sigma(l^*) + \sigma(k) \quad \textit{if } n \textit{ is odd}. \end{aligned}$$

Here $\sigma(k)$ denotes the signature of k, and is just the sum over all θ of $\sigma_\theta(k)$. The latter result is due to Shinohara [Sh].

2. Let K be the closed complement in S^3 of the solid torus V, and let $\tilde{K}$ be the infinite cyclic cover of K corresponding to the kernel of the Hurewicz

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map $\pi_1(K) \to H_1(K) \cong (t: \ )$. Then $H_1(\tilde{K})$ is a finitely-generated module over $\Lambda = \mathbf{Z}[t, t^{-1}]$, and there is a Blanchfield duality pairing

$$\langle\ ,\ \rangle: H_1(\tilde{K}) \times H_1(\tilde{K}) \to \Lambda_0/\Lambda,$$

where Λ_0 is the field of fractions of Λ. This pairing is Hermitian with respect to the conjugation defined by $t \mapsto t^{-1}$. It is also nonsingular.

Set $\Gamma = \mathbf{R}[t, t^{-1}]$, and pass to real coefficients: then we obtain a pairing

$$\langle\ ,\ \rangle: H_1(\tilde{K}; \mathbf{R}) \times H_1(\tilde{K}; \mathbf{R}) \to \Gamma_0/\Gamma.$$

Let $p(t)$ be a prime in Γ dividing $\Delta_k(t)$; and let V_p denote the $p(t)$-primary component of $H_1(\tilde{K}; \mathbf{R})$. As in **[K]**, V_p is orthogonal to V_q unless $(p(t)) = (q(t^{-1}))$. Moreover, if $p(t) = p(t^{-1})$, then V_p can be written as an orthogonal direct sum $V_p^1 \oplus \cdots \oplus V_p^m$, with V_p^r a free module over $\Gamma/(p^r)$. Let (x) denote the image in $H_p^r = V_p^r/pV_p^r$ of x in V_p^r; if $x,y \in V_p^r$, then we can define $[(x), (y)]_p^r = \langle p(t)^{r-1}x, y\rangle$.

Let $\varphi: \Gamma \to \Gamma/(p)$ be the quotient map; then defining $((x), (y))_p^r$ to be $\varphi(z)$, where $[(x), (y)]_p^r = z/p$, makes H_p^r into an Hermitian space over the field $\Gamma/(p) \cong \mathbf{C}$. Conjugation coincides with complex conjugation, as the roots of $p(t)$ lie on the complex unit circle. Let $\sigma_p^r(k)$ be the signature of the corresponding quadratic space, and let $\sigma_p(k)$ be the sum over odd r of the $\sigma_p^r(k)$.

It is shown in **[K]** that $\sigma_p(k) = \sigma_\theta(k)$, where $p(t) = t^{-1} - 2\cos\theta + t$, $0 < \theta < \pi$.

In passing, note that $\sigma_p(k)$ is an invariant of the cobordism class of k (see **[M₁]**); $\sigma_p^r(k)$ is an invariant of k, but not of its cobordism class **[L]**. As Milnor points out **[M₂]**, for r even the corresponding quadratic space is hyperbolic, and so $\sigma_\theta^r(k) = 0$.

3. Let N be a regular neighbourhood in T of l^*, L^* the closed complement of N in S^3, L' the closed complement of $f(N)$ in V, and L the closed complement of $f(N)$ in S^3. Then $L = L' \cup K$, and $L' \cap K = \partial V$. Passing to the infinite cyclic cover of L, $\tilde{L} = \tilde{L}' \cup \tilde{K}_1 \cup \cdots \cup \tilde{K}_n$, where the $\tilde{K}_r$ are disjoint copies of $\tilde{K}$, and $\tilde{L}' \cap \tilde{K}_r \cong S^1 \times \mathbf{R}$. We can number the $\tilde{K}_r$ so that the action of $(t: \ )$ on $\tilde{L}$ is given by $t\tilde{K}_r = \tilde{K}_{r+1}$, working modulo n.

It is implicit in the work of Seifert **[S]** that $H_1(\tilde{L})$ splits as a direct sum of Λ-modules, $H_1(\tilde{L}^*) \oplus H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_n)$. Furthermore, if $M(t)$ is a presentation matrix for $H_1(\tilde{K})$, then $M(t^n)$ is a presentation matrix for $H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_n)$: this too is easily deduced from **[S**; p. 32].

From the definition of the Blanchfield duality pairing **[B]**, it is clear that the direct sum above is orthogonal, and that $H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_n)$ splits as an orthogonal direct sum of $\mathbf{Z}$-modules $H_1(\tilde{K}_1) \oplus \cdots \oplus H_1(\tilde{K}_n)$.

4. Let $p(t) = t^{-1} - 2\cos\theta + t$, $0 < \theta < \pi$, be an irreducible factor of $\Delta_k(t)$, and let $p(t^n) = p_1(t) \cdots p_n(t)$ where $p_r(t) = t^{-1} - 2\cos\theta_r + t$, $0 < |\theta_r| < \pi$. Let $\tau = \exp(i\theta)$, and let $\tau_r = \exp(i\theta_r)$ be the root of $p_r(t)$ which is also an nth root of τ. Write $p^r(t) = p(t^n)/p_r(t)$.

Recall that V_p is the $p(t)$-primary component of $H_1(\tilde{K}; \mathbf{R})$. If we identify $H_1(\tilde{K}; \mathbf{R})$ with $H_1(\tilde{K}_1; \mathbf{R}) \subset H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_n; \mathbf{R})$, as a vector space, then clearly $(p^r(t))^N V_p$ is contained in V_{p_r}, the $p_r(t)$-primary component of $H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_n; \mathbf{R})$, for large N. Indeed, by considering a diagonal presentation matrix $M(t)$ for $H_1(\tilde{K}; \mathbf{R})$, and passing to $M(t^n)$, it is clear that $(p^r(t))^N V_p = V_{p_r}$.

Consider $x,y \in V_p$ as elements of $H_1(\tilde{K}; \mathbf{R})$; then $\langle x, y\rangle = \mu(t)/(p(t))^m$ say. Regarding x, y as elements of $H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_n; \mathbf{R})$, it follows from the definition of the duality pairing that $\langle x, y\rangle = \mu(t^n)/(p(t^n))^m$. Thus if $V_p = V_p^1 \oplus \cdots \oplus V_p^m$, an orthogonal direct sum in $H_1(\tilde{K}; \mathbf{R})$, we can take $V_{p_r}^s = (p^r(t))^N V_p^s$ to obtain an orthogonal direct sum $V_{p_r} = V_{p_r}^1 \oplus \cdots \oplus V_{p_r}^m$.

Let $x', y' \in H_{p_r}^s$, and choose $x,y \in V_p^s$ so that $x' = ((p^r(t))^N x)$, $y' = ((p^r(t))^N y)$. Then

$$\begin{aligned}[x', y']_{p_r}^s &= \langle p_r(t)^{s-1}(p^r(t))^N x, (p^r(t))^N y\rangle \\ &= (p^r(t))^{2N-s+1}\langle p(t^n)^{s-1}x, y\rangle \\ &= (p^r(t))^{2N-s+1}\mu(t^n)/p(t^n)\end{aligned}$$

where regarding x, y as elements of $H_1(\tilde{K}; \mathbf{R})$, the Blanchfield pairing of k gives $\langle p(t)^{s-1}x, y\rangle = \mu(t)/p(t)$. Thus

$$[x', y']_{p_r}^s = (p^r(t))^{2N-s}\mu(t^n)/p_r(t),$$

and so

$$(x', y')_{p_r}^s = (p^r(\tau_r))^{2N-s}\mu(\tau).$$

Of course, if we regard x, y as elements of $H_1(\tilde{K}; \mathbf{R})$, then in the Hermitian space H_p^s we have $((x), (y))_p^s = \mu(\tau)$. Thus it only remains for us to evaluate $p^r(\tau_r)$. Using L'Hôpital's rule, it is easy to see that

$$p^r(\tau_r) = \lim_{t\to\tau_r}\left(\frac{p(t^n)}{p_r(t)}\right) = n\frac{\mathcal{I}(\tau)}{\mathcal{I}(\tau_r)} = n\frac{\sin\theta}{\sin\theta_r},$$

where $\mathcal{I}(z)$ is the imaginary part of z.

Thus if s is odd, $H_{p_r}^s$ contributes $\sigma_\theta^s(k)$ sign $(n \sin\theta_r)$ to the signature of l; and if s is even, all the corresponding signatures are zero. This proves the theorem.

The corollary follows easily by considering the distribution of the τ_r around the unit circle.

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