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DEMENSION AND MEASURE

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ABSTRACT. We give a new characterization, based on Hausdorff measure, for the demension of a compact set in a euclidean space.

The *demension* dem X (dimension of embedding) of a compact set X in a euclidean space R^n was introduced by Štan'ko [4] to characterize the tameness of X in R^n. A good exposition of the theory has been given by Edwards [1]. There are several equivalent definitions for dem X (see [1, §1.2] and Štan'ko's original definition [4, §1.1]). In this note we give a *measure-theoretic* characterization of dem X, which is an analogue of the theorem of Szpilrajn (= Marczewski) concerning dim X [2, Theorem VII 1, p. 102].

We let $m_\alpha(X)$ denote the α-dimensional Hausdorff measure of a set $X \subset R^n$ and $\dim_H X$ the Hausdorff dimension of X. For definitions, see [2, pp. 103, 107].

THEOREM. *Let X be a compact set in R^n. Then* dem $X \leqslant k$ *if and only if there is a homeomorphism f: $R^n \to R^n$ such that $m_{k+1}(fX) = 0$. Moreover,* dem $X \leqslant \dim_H fX$ *for all homeomorphisms f: $R^n \to R^n$, and* dem $X = \dim_H fX$ *for some f.*

PROOF. If $m_{k+1}(fX) = 0$ for some homeomorphism f: $R^n \to R^n$, then dem $fX \leqslant k$ by [3, 6.15]. Since dem X is invariant under homeomorphisms of R^n [1, §1.1], this implies dem $X \leqslant k$.

To complete the proof of the theorem, it suffices to construct a compact set $P_n^k \subset R^n$ such that $\dim_H P_n^k \leqslant k$ and such that for every compact set $X \subset R^n$ with dem $X \leqslant k$ there is a homeomorphism f: $R^n \to R^n$ which maps X into P_n^k. We shall construct P_n^k by modifying the construction of Menger's compactum M_n^k.

We start with the unit cube $I^n = [0, 1]^n$. Subdivide I^n into 4^n cubes of side length $\frac{1}{4}$ and retain those which meet the k-faces of I^n. These will be called cubes of rank one. Proceeding inductively, assume that Q is a cube of rank $j - 1$. Subdivide Q into $2^{(j+1)n}$ equal cubes. Those which meet the k-faces of Q are called cubes of rank j. Let S_j be the union of all cubes of rank j. Then $P_n^k = \bigcap \{S_j | j \geqslant 1\}$.

We next show that $\dim_H P_n^k \leqslant k$, that is, $m_\alpha(P_n^k) = 0$ for every $\alpha > k$. Let r be the number of all k-faces of I^n. Since each cube of rank $j - 1$ contains at most $2^{(j+1)k}r$ cubes of rank j, there are at most $4^k 8^k \ldots 2^{(j+1)k} r^j = 2^{j(j+3)k/2} r^j$

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cubes of rank j. The side length of such a cube is $2^{-j(j+3)/2}$. Hence these cubes form a cover $\{Q_1, \ldots, Q_s\}$ of P_n^k such that

$$\sum_{i=1}^{s} d(Q_i)^\alpha \leqslant (2^{(k-\alpha)(j+3)/2} r)^j n^{\alpha/2}.$$

Since $\alpha > k$, the right-hand side tends to zero as $j \to \infty$. Thus $m_\alpha(P_n^k) = 0$.

Suppose that X is compact in R^n and dem $X \leqslant k$. By a result of Štan'ko [5] (see also Edwards [1, §1.2]), there is a homeomorphism f: $R^n \to R^n$ (in fact, an isotopy of R^n with compact support) which carries X into Menger's compactum M_n^k. It is easy to see that M_n^k can be replaced by P_n^k in the proof of this result. □

REMARKS. There is an isotopy version of the above result, since the map f in the proof can be obtained by an isotopy of R^n with compact support. In fact, the isotopy can be chosen to be arbitrarily small by using a stack of small copies of P_n^k (cf. Edwards [1, pp. 208–209]).

The result can be extended to closed subsets of Lipschitz manifolds (cf. [1, §2]).

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