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AN ADDENDUM TO "ON GENERATING DISTRIBUTIVE SUBLATTICES OF ORTHOMODULAR LATTICES"

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ABSTRACT. This addendum provides the details of a computation not presented in [1] but needed for a complete proof that Foulis-Holland sets generate distributive sublattices.

In [1] I called a nonempty subset S of an orthomodular lattice a *Foulis-Holland set* in case whenever x, y and z are distinct elements of S one of them commutes with the other two. I presented a proof that Foulis-Holland sets generate distributive sublattices. The purpose of this note is to provide a detailed proof that the function ψ defined in Lemma 2.2 of [1] is indeed onto.

Throughout this paper let $S = \{s_1, \ldots, s_n, t_1, \ldots, t_n\}$ be a finite nonempty subset of an orthomodular lattice L such that $s_i \in C(S \setminus \{t_i\})$ and $t_i \in C(S \setminus \{s_i\})$, for $i = 1, \ldots, n$, and let

$$A_S = \{x_1 \wedge \cdots \wedge x_n | x_i \in \{s_i, t_i\}\} \setminus \{0\}.$$

Lemma 2.2 of [1] states that the power set $\mathcal{P}(A_S)$ of A_S is isomorphic to the sublattice $\langle S \rangle$ of L generated by S in case, for each $i = 1, 2, \ldots, n$, s_i and t_i are complements in L. The proof proceeds by defining $\psi: \mathcal{P}(A_S) \to \langle S \rangle$ by the rule $\psi(M) = \bigvee M$ for $M \subseteq A_S$. A computation shows that $M \subseteq N$ if and only if $\psi(M) < \psi(N)$. A shorter computation shows that $S \subseteq \text{image}(\psi)$ from which it is claimed that ψ is onto (and therefore a lattice isomorphism). What is missing is a proof that image(ψ) is a sublattice of $\langle S \rangle$ (or equivalently of L). Clearly ψ preserves joins. But it is not clear that ψ preserves meets. This fact is needed to get from $S \subseteq \text{image}(\psi)$ to $\langle S \rangle \subseteq \text{image}(\psi)$. I am indebted to Professor M. F. Janowitz for this observation.

That ψ preserves meets is the content of the following proposition. We begin by reviewing some notation and making some observations.

For $M \subseteq A_S$, define $\delta(M) = \{x_2 \wedge \cdots \wedge x_n |$ for some $x_1 \in \{s_1, t_1\}$, $x_1 \wedge \cdots \wedge x_n \in M\}$ and for $y_1 \in \{s_1, t_1\}$ let

$$M_{y_1} = \{x_1 \wedge \cdots \wedge x_n \in M | y_1 = x_1\}.$$

Assume that s_i and t_i are complements in L, $i = 1,2, \ldots, n$.

LEMMA. *If $M,N \subseteq A_S$ and $x \in \{s_1, t_1\}$, then*
(L1) $\bigvee M = \bigvee M_{s_1} \vee \bigvee M_{t_1}$,

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(L2) $\delta(M_x) \cap \delta(N_x) = \delta((M \cap N)_x)$,
(L3) $\bigvee M_x = x \wedge (\bigvee \delta(M_x))$,
(L4) $\bigvee M = [s_1 \vee \bigvee \delta(M_{t_1})] \wedge [t_i \vee \bigvee \delta(M_{s_1})] \wedge [\bigvee \delta(M)]$,
(L5) $\bigvee M > \bigvee \delta(M_{s_1}) \wedge \bigvee \delta(M_{t_1})$,
(L6) $\bigvee(M \cap N) < \bigvee (\delta(M) \cap \delta(N))$.

Proof. (L1) and (L2) follow from the definitions. (L3) is simply an application of the Foulis-Holland Theorem. By (L1), (L3) and the Marsden-Herman Lemma

$$\begin{aligned}\bigvee M &= \bigvee (M_{s_1}) \vee \bigvee (M_{t_1}) = \left[s_1 \wedge (\bigvee \delta(M_{s_1}))\right] \vee \left[t_1 \wedge (\bigvee \delta(M_{t_1}))\right]\\ &= (s_1 \vee t_1) \wedge \left[s_1 \vee \bigvee \delta(M_{t_1})\right] \wedge \left[t_1 \vee \bigvee \delta(M_{s_1})\right]\\ &\quad \wedge \left[\bigvee \delta(M_{s_1}) \vee \bigvee \delta(M_{t_1})\right].\end{aligned}$$

(L4) now follows from the fact that s_1 and t_1 are complements and $\delta M = \delta(M_{s_1}) \cup \delta(M_{t_1})$. (L5) follows immediately from (L4). Finally, (L6) follows from the fact that, for each $c \in M \cap N$, $\delta(c) \in \delta(M) \cap \delta(N)$ and $c < \delta(c)$.

Proposition. *For* $M,N \subseteq A_S$, $(\bigvee M) \wedge (\bigvee N) = \bigvee (M \cap N)$.

Proof. Let $m = (\bigvee M) \wedge (\bigvee N)$. If $n = 1$, the result is obvious. Assume the result true for all $k < n$. By (L1), (L3) and (L4) of the lemma and the Foulis-Holland Theorem

$$\begin{aligned}m &= \left[s_1 \vee \bigvee \delta(M_{t_1})\right] \wedge \left[s_1 \vee \bigvee \delta(N_{t_1})\right] \wedge \left[t_1 \vee \bigvee \delta(M_{s_1})\right]\\ &\quad \wedge \left[t_1 \vee \bigvee \delta(N_{s_1})\right] \wedge \left[\bigvee \delta(M)\right] \wedge \left[\bigvee \delta(N)\right]\\ &= \left[s_1 \vee ((\bigvee \delta(M_{t_1})) \wedge (\bigvee \delta(N_{t_1})))\right]\\ &\quad \wedge \left[t_1 \vee ((\bigvee \delta(M_{s_1})) \wedge (\bigvee \delta(N_{s_1})))\right] \wedge \left[\bigvee \delta(M)\right] \wedge \left[\bigvee \delta(N)\right].\end{aligned}$$

Invoking the induction hypothesis we have

$$\begin{aligned}m &= \left[s_1 \vee \bigvee (\delta(M_{t_1}) \cap \delta(N_{t_1}))\right] \wedge \left[t_1 \vee \bigvee (\delta(M_{s_1}) \cap \delta(N_{s_1}))\right]\\ &\quad \wedge \left[\bigvee (\delta(M) \cap \delta(N))\right].\end{aligned}$$

By (L2), the Marsden-Herman Lemma and the fact that $s_1 \wedge t_1 = 0$, we have

$$\begin{aligned}m &= \left(\left[s_1 \wedge (\bigvee \delta((M \cap N)_{s_1}))\right] \vee \left[t_1 \wedge (\bigvee \delta((M \cap N)_{t_1}))\right]\right.\\ &\qquad \left.\vee \left[(\bigvee \delta((M \cap N)_{t_1})) \wedge (\bigvee \delta((M \cap N)_{s_1}))\right]\right)\\ &\quad \wedge \left[\bigvee (\delta A \cap \delta B)\right].\end{aligned}$$

By (L3) applied to $(M \cap N)_x$, the first two terms reduce to $[\bigvee (M \cap N)_{s_1}] \vee [\bigvee (M \cap N)_{t_1}]$ which by (L1) equals $\bigvee (M \cap N)$. Thus

$$\begin{aligned}m &= \left(\left[\bigvee (M \cap N)\right] \vee \left[\bigvee \delta((M \cap N)_{s_1}) \cap \delta((M \cap N)_{t_1})\right]\right)\\ &\quad \wedge \left[\bigvee (\delta(M) \cap \delta(N))\right]\\ &= \left[\bigvee (M \cap N)\right] \wedge \left[\bigvee (\delta(M) \cap \delta(N))\right] = \bigvee (M \cap N)\end{aligned}$$

where the second equality follows from (L2) applied to $M \cap N$ rather than M and the last equality follows from (L3). The proposition is proved.

References

1. R. J. Greechie, *On generating distributive sublattices of orthomodular lattices*, Proc. Amer. Math. Soc. 67 (1977), 17–22.

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