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LINEAR MAPS OF $\mathcal{C}^*$-ALGEBRAS PRESERVING THE ABSOLUTE VALUE

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ABSTRACT. In order that a linear map of $\mathcal{C}^*$-algebras ϕ: $\mathcal{A} \to \mathcal{B}$ preserve absolute values, it is necessary and sufficient that it be 2-positive and preserve zero products of positive elements: if x and y are positive in $\mathcal{A}$, with $xy = 0$, then $\phi(x)\phi(y) = 0$.

The generalized Schwarz inequalities of Kadison and Choi are extended to the nonunital case.

1. Introduction. In [4], linear maps ϕ of $\mathcal{C}^*$-algebras which preserve the absolute value were characterized as $*$-homomorphisms ψ followed by a map $x \to bx = b^{1/2}xb^{1/2}$, where b is a positive element centralizing the image of ψ. This sequel to [4] has as its principal purpose the exploration of other characterizations of these maps. The main result is that a linear map ϕ: $\mathcal{A} \to \mathcal{B}$ of $\mathcal{C}^*$-algebras preserves the absolute value if and only if ϕ is 2-positive and preserves zero products of positive elements: if x and y are positive in $\mathcal{A}$, and $xy = 0$, then $\phi(x)\phi(y) = 0$ in $\mathcal{B}$. This is the main burden of Theorem 2. 2-positivity is relaxed to positivity in the presence of lots of projections in $\mathcal{A}$ (Theorem 1).

As in [4], one of our principal tools is the result of S. Sherman [8] to the effect that if $\mathcal{A}$ is a $\mathcal{C}^*$-algebra, the second conjugate (or double-dual) space $(\mathcal{A}^d)^d$ of the underlying Banach space $\mathcal{A}$ has a natural structure of W^*-algebra in which $\mathcal{A}$ is σ-weakly dense. $\mathcal{A}^{dd}$ can be represented concretely as the σ-weak closure of $\pi(\mathcal{A})$, if π is the *universal representation* of $\mathcal{A}$, the direct sum $\bigoplus_\sigma \pi_\sigma$ of the cyclic representations π_σ arising from the states (normalized positive linear functionals) σ of $\mathcal{A}$ by the Gel'fand-Neĭmark-Segal construction. The Hilbert space underlying π we call the *universal representation space* of $\mathcal{A}$. For a swift and complete account of this, see Kadison's article [6].

Roughly, our proofs proceed by showing that our hypotheses persist from ϕ: $\mathcal{A} \to \mathcal{B}$ to the map $(\phi^d)^d = \phi^{dd}$: $\mathcal{A}^{dd} \to \mathcal{B}^{dd}$ (second transpose map), and that $\phi^{dd}(I)$ which for simplicity we call $\phi(I)$, or b, lies in the centre of $\phi(\mathcal{A}^{dd})$. We then compose ϕ^{dd} with $x \to b^{-1}x = b^{-1/2}xb^{-1/2}$, a routine made precise and explicit in [4], use known results, including spectral theory, to establish the desired properties of the composed map ψ, and climb back down.

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Other principal tools are the main results of Kaplansky's paper [**7**], both the density theorem and the strong continuity of the continuous functional calculus, and Choi's generalized Schwarz inequality [**1**].

By-products of the investigation include nonunital versions of the Choi and Kadison generalized Schwarz inequalities (Corollaries 1 and 2), of Choi's result that a 2-positive unital Jordan map of $\mathcal{C}^*$-algebras is a $*$-homomorphism (Corollary 6), and of Kadison's result that a unital linear map of $\mathcal{C}^*$-algebras preserving absolute values on self-adjoint elements is a Jordan map (Corollary 7).

2. Notations and definitions; statement of main results. For generalities on $\mathcal{C}^*$-algebras and W^*-algebras (von Neumann algebras), see the books of J. Dixmier [**2**], [**3**].

If $\mathcal{A}$ is a $\mathcal{C}^*$-algebra, $\mathcal{A}^+ = \{x^*x: x \in \mathcal{A}\}$ is a closed, convex, proper cone, linearly spanning $\mathcal{A}$. Every element a of $\mathcal{A}^+$ has a unique square root $a^{1/2}$ in $\mathcal{A}^+$. If $x \in \mathcal{A}$, $|x| = (x^*x)^{1/2}$ is the *absolute value* of x. A linear map ϕ: $\mathcal{A} \to \mathcal{B}$ is *positive* if $\phi(\mathcal{A}^+) \subset \mathcal{B}^+$, and 2-*positive* if the map $\phi \otimes \mathrm{id}_2$ is positive on the $\mathcal{C}^*$-algebra $\mathcal{A} \otimes M_2(C)$ to $\mathcal{B} \otimes M_2(C)$. Here $M_2(C)$ is the $\mathcal{C}^*$-algebra of 2×2 complex matrices [**9**]. The $\mathcal{C}^*$-algebra $\mathcal{A}$ is *unital* if $\mathcal{A}$ has a unit element $I_{\mathcal{A}}$ or I. ϕ is a *Jordan map* (C^*-homomorphism) if $\phi(x^2) = \phi(x)^2$ for all (selfadjoint) x in $\mathcal{A}$. If $\mathcal{A} \subset L(\mathfrak{H})$, the $\mathcal{C}^*$-algebra of all bounded linear operators on the Hilbert space $\mathfrak{H}$, and if $\eta \in \mathfrak{H}$, then ω_η is the positive linear functional $x \to \langle x\eta, \eta\rangle$ on $\mathcal{A}$.

DEFINITION. A linear map ϕ: $\mathcal{A} \to \mathcal{B}$ of $\mathcal{C}^*$-algebras will be called *disjoint* if $xy = 0$ in $\mathcal{A}$ implies $\phi(x)\phi(y) = 0$ in $\mathcal{B}$.

THEOREM 1. *A* 2-*positive, disjoint linear map of $\mathcal{C}^*$-algebras preserves absolute values.*

If the domain algebra is AW^* or approximately finite, "2-positive" can be replaced by "positive".

THEOREM 2. *For a linear map ϕ: $\mathcal{A} \to \mathcal{B}$ of $\mathcal{C}^*$-algebras, the following conditions are equivalent:*

(i) *ϕ preserves absolute values;*

(ii) *ϕ is positive, and $\phi(I)\phi(a_1a_2) = \phi(a_1)\phi(a_2)$ for all $a_1, a_2 \in \mathcal{A}$;*

(iii) *ϕ is 2-positive and disjoint;*

(iii)$'$ *ϕ is 2-positive, and disjoint on positive elements.*

In (ii) and hereafter, $\phi(I)$ has the interpretation $\phi^{\mathrm{dd}}(I) \in \mathcal{B}^{\mathrm{dd}}$.

3. Details, proofs. The proofs of the theorems will follow a series of lemmas, some of independent interest.

LEMMA 1. *Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra in which the linear span of the projections is norm-dense. Let $\mathcal{B}$ be a unital $\mathcal{C}^*$-algebra, and let ϕ: $\mathcal{A} \to \mathcal{B}$ be positive, unital and disjoint. Then ϕ is $*$-homomorphic.*

PROOF. Since ϕ is a positive unital map, it is selfadjoint. If e is a projection in $\mathcal{Q}$, and $a \in \mathcal{Q}$, write

$$a = ae + a(1 - e)$$

and

$$\phi(a) = \phi(ae) + \phi(a(1 - e)),$$

and since $[a(1 - e)]e = 0$,

$$\phi(a)\phi(e) = \phi(ae)\phi(e),$$

while since $[ae](1 - e) = 0$,

$$\phi(ae)(1 - \phi(e)) = 0$$

or

$$\phi(ae)\phi(e) = \phi(ae).$$

Thus, $\phi(ae) = \phi(a)\phi(e)$, for all $a \in \mathcal{Q}$ and projections $e \in \mathcal{Q}$, but then since the linear span of the projection is dense in $\mathcal{Q}$,

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all } a, b \in \mathcal{Q}.$$

$\therefore$ ϕ is $*$-homomorphic. $\square$

LEMMA 2. *For a positive linear map ϕ: $\mathcal{Q} \to \mathcal{B}$ of $\mathcal{C}^*$-algebras, the following are equivalent*:

(i) ϕ *preserves* $|\cdot|$ *on selfadjoint elements*;

(ii) ϕ *is disjoint on selfadjoint elements.*

PROOF. (i) $\Rightarrow$ (ii). Suppose a_1, a_2 are selfadjoint in $\mathcal{Q}$, and $a_1a_2 = 0$. Then since $a_1a_2^* = 0$,

$$a_1^*a_1a_2^*a_2 = 0, \qquad |a_1|\,|a_2| = 0,$$

and

$$||a_1| - |a_2|| = |a_1| + |a_2|.$$

But then

$$|\phi(|a_1| - |a_2|)| = |\phi(|a_1|) - \phi(|a_2|)| = \phi(|a_1|) + \phi(|a_2|),$$

so $\phi(|a_1|)\phi(|a_2|) = 0$, that is, $|\phi(a_1)|\,|\phi(a_2)| = 0$, so $\phi(a_1)\phi(a_2) = 0$.

(ii) $\Rightarrow$ (i). If $a = a^* \in \mathcal{Q}$, let $a = a^+ - a^-$ be its canonical decomposition with $a^+ > 0$, $a^- > 0$, $a^+a^- = 0$. Then $\phi(a) = \phi(a^+) - \phi(a^-)$ with $\phi(a^+) > 0$, $\phi(a^-) > 0$, $\phi(a^+)\phi(a^-) = 0$, so $\phi(a^+) = \phi(a)^+$, $\phi(a^-) = \phi(a)^-$, and finally,

$$|\phi(a)| = \phi(a)^+ + \phi(a)^- = \phi(a^+ + a^-) = \phi(|a|). \quad \square$$

LEMMA 3. *If ϕ: $\mathcal{Q} \to \mathcal{B}$ is a 2-positive, linear map of $\mathcal{C}^*$-algebras, then ϕ^{dd}: $\mathcal{Q}^{\mathrm{dd}} \to \mathcal{B}^{\mathrm{dd}}$ is 2-positive, and satisfies $\phi^{\mathrm{dd}}(x)^*\phi^{\mathrm{dd}}(x) < \|\phi\|\phi^{\mathrm{dd}}(x^*x)$ for all $x \in \mathcal{Q}^{\mathrm{dd}}$.*

PROOF. ϕ is positive, so bounded; normalizing, we suppose it contractive. Then ϕ^{dd} is a positive contraction. Put $b = \phi^{\mathrm{dd}}(I)$, and let P be the support projection of b. Then b^{-1} exists as a (usually unbounded) positive, selfadjoint operator affiliated with $P\mathfrak{B}^{dd}P$, and $\psi(x) = b^{-1/2}\phi^{dd}(x)b^{-1/2}$ is well defined in $P\mathfrak{B}^{dd}P = \mathfrak{D} \subset \mathfrak{B}^{dd}$ for $x \in \mathfrak{A}^{dd}$. Moreover, ψ so defined is positive and unital on $\mathfrak{A}^{dd}$, and in fact is 2-positive since if id_2 and I_2, are, respectively, the identity mapping and the identity element of $M_2(\mathbf{C})$, we have

$$\begin{aligned}\psi \otimes \mathrm{id}_2 &= \left(b^{-1/2}\phi^{dd}(\cdot)b^{-1/2}\right) \otimes \mathrm{id}_2\\ &= \left(b^{-1/2} \otimes I_2\right)\left(\phi^{dd}(\cdot) \otimes \mathrm{id}_2\right)\left(b^{-1/2} \otimes I_2\right),\end{aligned}$$

while ϕ^{dd} is 2-positive because $\phi^{dd} \otimes \mathrm{id}_2 = (\phi \otimes \mathrm{id}_2)^{dd}$ is σ-weakly continuous on $\mathfrak{A}^{dd} \otimes M_2(\mathbf{C}) = (\mathfrak{A} \otimes M_2(\mathbf{C}))^{dd}$, and $\phi \otimes \mathrm{id}_2$ is positive. Now Choi's generalized Schwarz inequality [**1**, Corollary 2.8] applies, so that for $x \in \mathfrak{A}^{dd}$, we have $\psi(x)^*\psi(x) \leqslant \psi(x^*x)$, or writing $\tilde{\phi}$ for ϕ^{dd}, $b^{-1/2}\tilde{\phi}(x)^*b^{-1}\tilde{\phi}(x)b^{-1/2} \leqslant b^{-1/2}\tilde{\phi}(x^*x)b^{-1/2}$, whence $\tilde{\phi}(x)^*b^{-1}\tilde{\phi}(x) \leqslant \tilde{\phi}(x^*x)$. Now since $0 \leqslant b \leqslant I$ in $\mathfrak{B}$,

$$\begin{aligned}\tilde{\phi}(x)^*\tilde{\phi}(x) &= \operatorname*{st.lim}_n \tilde{\phi}(x)^*b\left(b + \frac{1}{n} I\right)^{-1}\tilde{\phi}(x)\\ &\leqslant \operatorname*{st.lim}_n \tilde{\phi}(x)^*\left(b + \frac{1}{n} I\right)^{-1}\tilde{\phi}(x) = \tilde{\phi}(x)^*b^{-1}\tilde{\phi}(x)\\ &\leqslant \tilde{\phi}(x^*x),\end{aligned}$$

for $x \in \mathfrak{A}^{dd}$. Returning to the original, possibly noncontractive ϕ, we have the inequality claimed in the lemma. □

The next two corollaries are nonunital generalizations of the results cited. Corollary 2, in addition to removing the unital restriction, treats not only selfadjoint but normal elements, as does Størmer's Theorem 3.1 in [**10**].

COROLLARY 1 (CHOI'S GENERALIZED SCHWARZ INEQUALITY). *If* ϕ: $\mathfrak{A} \to \mathfrak{B}$ *is a* 2-*positive linear map of* $\mathcal{C}^*$-*algebras*, $\phi(x)^*\phi(x) \leqslant \|\phi\|\phi(x^*x)$ *for all* $x \in \mathfrak{A}$.

COROLLARY 2 (KADISON'S GENERALIZED SCHWARZ INEQUALITY). *If* ϕ: $\mathfrak{A} \to \mathfrak{B}$ *is a positive linear map of* $\mathcal{C}^*$-*algebras*, $|\phi(x)|^2 \leqslant \|\phi\|\phi(|x|^2)$ *for all normal* $x \in \mathfrak{A}$.

PROOF. The restriction of ϕ to a commutative sub-$\mathcal{C}^*$-algebra of $\mathfrak{A}$ is completely positive, by Stinespring [**9**, Theorem 4]. Since every normal $x \in \mathfrak{A}$ is contained in such a subalgebra, Corollary 2 now follows from Corollary 1. □

COROLLARY 3. *If* ϕ: $\mathfrak{A} \to \mathfrak{B}$ *is a* 2-*positive linear map of* $\mathcal{C}^*$-*algebras*, ϕ^{dd} *is strongly continuous.*

PROOF. If η is a vector in the universal representation space of $\mathfrak{B}$, we have, if, as we may assume, ϕ is contractive,

$$\|\phi^{dd}(x)\eta\|^2 = \langle \phi^{dd}(x)^*\phi^{dd}(x)\eta, \eta\rangle < \langle \phi^{dd}(x^*x)\eta, \eta\rangle$$
$$= \omega_\eta(\phi^{dd}(x^*x)) = (\phi^d(\omega_\eta))(x^*x) = \|x\xi\|^2$$

for some ξ in the universal representation space of $\mathcal{A}$ independent of $x \in \mathcal{A}^{dd}$. $\square$

LEMMA 4. *If ϕ is 2-positive, and disjoint on positive elements, ϕ is disjoint.*

PROOF.

$$a_1a_2^* = 0 \Leftrightarrow a_1^*a_1a_2^*a_2 = 0 \Leftrightarrow |a_1|^2|a_2|^2 = 0$$
$$\Rightarrow \phi(|a_1|^2)\phi(|a_2|^2) = 0.$$

Since by Lemma 3,

$$\|\phi\|\phi(|a_i|^2) > |\phi(a_i)|^2, \qquad |\phi(a_1)|^2|\phi(a_2)|^2 = 0,$$

so $\phi(a_1)\phi(a_2^*) = 0$. $\square$

COROLLARY 4. *A 2-positive, Jordan map is disjoint.*

PROOF. Apply Lemmas 2 and 4. $\square$

COROLLARY 5 (CHOI [**1**, Corollary 3.2]). *A 2-positive unital Jordan map of $\mathcal{C}^*$-algebras is a $*$-homomorphism.*

PROOF. If ϕ: $\mathcal{A} \to \mathcal{B}$ is 2-positive unital and Jordan, so is ϕ^{dd}: $\mathcal{A}^{dd} \to \mathcal{B}^{dd}$, by Lemma 3 and Corollary 3; by the previous corollary, ϕ^{dd} is disjoint; by Lemma 1, ϕ^{dd} is $*$-homomorphic: Therefore, so is its restriction ϕ. $\square$

REMARK. This proof is neither simpler than Choi's original proof, nor independent of the main results of his paper [**1**], but see Corollary 6.

PROOF OF THEOREM 1. Let ϕ: $\mathcal{A} \to \mathcal{B}$ be a 2-positive, disjoint linear map of $\mathcal{C}^*$-algebras. Then by Corollary 3, ϕ^{dd} is strongly continuous, while, by Lemma 2, ϕ preserves $|\cdot|$ on selfadjoint elements. By Kaplansky's density theorem and the strong continuity of $|\cdot|$ on bounded sets of selfadjoint operators, we see that ϕ^{dd} preserves $|\cdot|$ on selfadjoint elements of $\mathcal{A}^{dd}$, so that (again Lemma 2) ϕ^{dd} is disjoint on the selfadjoint part of $\mathcal{A}^{dd}$. Especially, if e_i ($i = 1, 2$) are projections in $\mathcal{A}^{dd}$, and $e_1e_2 = 0$, then $\phi(e_1)\phi(e_2) = 0$. (Where no confusion can result, we write ϕ instead of ϕ^{dd}.) Then if $a \in \mathcal{A}^{dd}$, and $a = \sum_1^n\lambda_je_j$, with $\lambda_j \in \mathbf{C}$, the e_j pairwise orthogonal projections, $\phi(a) = \sum_1^n\lambda_j\phi(e_j)$, with the $\phi(e_j)$ disjoint and positive, so

$$|\phi(a)| = \sum |\lambda_j|\phi(e_j) = \phi\left(\sum |\lambda_j|e_j\right) = \phi(|a|).$$

Thus by spectral theory, ϕ^{dd} preserves $|\cdot|$ on normal elements, so on (unital) commutative $*$-subalgebras. From the first part of the proof of Theorem 1 of [**4**], we can conclude that $b = \phi^{dd}(I)$ commutes with each normal element of $\phi^{dd}(\mathcal{A}^{dd})$, so with all of $\phi^{dd}(\mathcal{A}^{dd})$, and that ψ defined on $\mathcal{A}^{dd}$ by $\psi(a) = b^{-1}\cdot\phi^{dd}(a)$ is $*$-homomorphic on commutative $*$-subalgebras of $\mathcal{A}^{dd}$, so is a Jordan homomorphism.

Since ψ is by Lemma 3 a 2-positive map as well as Jordan-homomorphic, and unital to Supp b $\mathfrak{B}^{dd}$Supp b, Corollary 5 shows it $*$-homomorphic. Then $\phi = b\psi = b^{1/2}\psi b^{1/2}$ preserves absolute values. This proves the first statement of Theorem 1.

If $\mathcal{A}$ is unital, and the set of linear combinations of orthogonal families of projections in $\mathcal{A}$ is norm-dense in $\mathcal{A}$, and if ϕ: $\mathcal{A} \to \mathfrak{B}$ is positive and disjoint, then we need not lift the argument above to $\mathcal{A}^{dd}$, but argue directly in $\mathcal{A}$ as above that $b = \phi(I)$ centralizes $\phi(\mathcal{A})$ in $\mathfrak{B}$, so

$$\psi = b^{-1}\phi\colon \mathcal{A} \to \text{Supp } b\ \mathfrak{B}^{dd} \text{ Supp } b$$

(see [4, Theorem 2]), is positive, unital and disjoint. Now Lemma 1 shows that ψ is $*$-homomorphic, so $\phi = b\psi$ preserves absolute values. This proves the second statement of Theorem 1. □

Lemma 5. *If $\mathcal{A}$ is a unital ring, $\mathfrak{B}$ a ring, and ϕ: $\mathcal{A} \to \mathfrak{B}$ is an additive map satisfying $\phi(I)\phi(x^2) = \phi(x)^2$ for all x in $\mathcal{A}$, then $\phi(I)$ centralizes $\phi(\mathcal{A})$.*

Proof.

$$\begin{aligned}\phi(I)\phi((I+x)^2) &= \phi(I)\phi(I + 2x + x^2) = \phi(I)(\phi(I) + 2\phi(x)) + \phi(x^2)\\ &= \phi(I)(\phi(I) + 2\phi(x)) + \phi(x)^2\\ &= \phi(I)^2 + 2\phi(I)\phi(x) + \phi(x)^2;\end{aligned}$$

but this is

$$(\phi(I) + \phi(x))^2 = \phi(I)^2 + \phi(I)\phi(x) + \phi(x)\phi(I) + \phi(x)^2,$$

so $\phi(I)\phi(x) = \phi(x)\phi(I)$ for all $x \in \mathcal{A}$. □

Proof of Theorem 2. That (i) $\Rightarrow$ (ii) was established in Theorem 2 of [4].

(ii) $\Rightarrow$ (i). Because ϕ^{dd} is σ-weakly continuous, and multiplication is separately continuous in the σ-weak topology on $\mathcal{A}^{dd}$ and $\mathfrak{B}^{dd}$, the identity (ii) persists for ϕ^{dd}, with $a_1, a_2 \in \mathcal{A}^{dd}$. Then Lemma 5 applies, and $b = \phi(I)$ centralizes $\phi^{dd}(\mathcal{A}^{dd})$. It then follows from (ii) that $\psi = b^{-1}\phi^{dd}$ is $*$-homomorphic on $\mathcal{A}^{dd}$ so $\phi^{dd} = b\psi$ preserves absolute values, as does its restriction ϕ.

(i) $\Rightarrow$ (iii). That (i) $\Rightarrow \phi$ completely positive and disjoint follows from Theorem 1 of [4].

(iii) $\Rightarrow$ (i). This is Theorem 1.

(iii) $\Rightarrow$ (iii)$'$. Trivial.

(iii)$'$ $\Rightarrow$ (iii). This is Lemma 4. □

Corollary 6 (The nonunital version of Corollary 5). *A 2-positive Jordan map of $\mathcal{C}^*$-algebras is a $*$-homomorphism.*

Proof. Let ϕ: $\mathcal{A} \to \mathfrak{B}$ be such a map. Then by Corollary 4, ϕ is 2-positive and disjoint, so, by Theorem 2, ϕ preserves absolute values. Then, by Theorem 2 of [4], $\phi^{dd} = \phi(I)\psi$, where ψ is a unital $*$-homomorphism of $\mathcal{A}^{dd} \to$ Supp $\phi(I)\mathfrak{B}^{dd}$Supp $\phi(I)$. But ϕ^{dd} is a Jordan map, so $\phi(I) = \phi^{dd}(I)$ is a projection: namely, Supp $\phi(I)$. Thus $\phi^{\mathrm{dd}} = \psi$, and its restriction ϕ is $*$-homomorphic. $\square$

COROLLARY 7 (NONUNITAL VERSION OF [5, THEOREM 6]). *If ϕ: $\mathcal{A} \to \mathcal{B}$ is a linear map of $\mathcal{C}^*$-algebras which preserves absolute values on selfadjoint elements, then there exist a unique Jordan map ψ: $\mathcal{A} \to \mathcal{B}^{\mathrm{dd}}$ and a unique positive element b of $\mathcal{B}^{\mathrm{dd}}$ supported on $\overline{\bigcup_{a\in\mathcal{A}} \text{range}\, \phi(a)}$ and centralizing $\phi(\mathcal{A})$ such that $\phi(x) = b\psi(x)$ for all $x \in \mathcal{A}$.*

PROOF. If ϕ: $\mathcal{A} \to \mathcal{B}$ preserves $|\cdot|$ on selfadjoint elements, so does ϕ^{dd}: $\mathcal{A}^{\mathrm{dd}} \to \mathcal{B}^{\mathrm{dd}}$, as in the proof of Theorem 1, so by Lemma 2, ϕ^{dd} preserves disjointness on selfadjoint elements. Let $\mathfrak{M}$ be a maximal commutative $*$-subalgebra of $\mathcal{A}^{\mathrm{dd}}$. Then $I \in \mathfrak{M}$. By [9, Theorem 4], the restriction of ϕ^{dd} to $\mathfrak{M}$ is completely positive, so 2-positive; therefore it preserves absolute values, by Lemma 4 and Theorem 1. Now the first part of the proof of Theorem 1 in [4] shows that $\phi(I)$ centralizes $\phi^{\mathrm{dd}}(\mathfrak{M})$, so $\phi(I)$ commutes with $\phi^{\mathrm{dd}}(u)$ for all unitary u in $\mathcal{A}^{\mathrm{dd}}$, hence with all $\phi^{\mathrm{dd}}(x)$, $x \in \mathcal{A}^{\mathrm{dd}}$. Now with $b = \phi(I)$ and $P = \text{Supp}\, b$,

$$\psi = b^{-1}\phi^{\mathrm{dd}} = b^{-1/2}\phi^{\mathrm{dd}}(\cdot)b^{-1/2},$$

mapping $\mathcal{A}^{\mathrm{dd}}$ unitally into $P\mathcal{B}^{\mathrm{dd}}P$ satisfies the hypotheses of Theorem 5 of [5], so is a Jordan map, and $\phi = b\psi$. This proves the existence claim.

The uniqueness claim can be proved following the uniqueness proof of [4], Theorem 2, or can be inferred from [4], Theorem 2 by restriction to commutative subsystems. $\square$

REMARK. The transposition map on $M_2(\mathbf{C})$ is unital, positive, and disjoint on positive elements, but is not disjoint. This shows that "2-positive" cannot be weakened to "positive" in (iii)′.

Problem. Can "2-positive" be replaced by "positive" in the first part of Theorem 1?

Finally, we note a further, easily proved relation between the properties "Jordan" and "$|\cdot|$-preserving" for linear maps: A Jordan homomorphism of $\mathcal{C}^*$-algebras which preserves absolute values is a $*$-homomorphism. In fact, if ψ: $\mathcal{A} \to \mathcal{B}$ is such a map,

$$\psi(a^*a) = \psi(|a|^2) = \psi(|a|)^2 = |\psi(a)|^2 = \psi(a^*)\psi(a).$$

A polarization completes the proof.

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