PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 1, October 1979

ADDENDUM TO "RESIDUAL LINEARITY FOR CERTAIN NILPOTENT GROUPS"

P. MENAL

ABSTRACT. In this note we offer an example of a residually $\mathbf{Q}$-linear, but not residually finite, nilpotent group whose center is $\mathbf{Z} \oplus \mathbf{Z}$.

This work rounds off some things which have been considered in [**1**].

Probably the best known result on residual finiteness for linear groups is Mal'cev's Theorem which says that a finitely generated linear group, over a commutative field, is residually finite. This result gives, in particular, that for finitely generated groups, residual finiteness and residual linearity are equivalent. Suppose now that G is a nilpotent linear group. Then it is quite easy to prove that G is finitely generated if its center is so [**1**, Lemma 1]. Hence we see that a nilpotent linear group with finitely generated center is residually finite. Thus it seems sensible to ask: Is any residually linear nilpotent group with finitely generated center always residually finite? For nilpotent groups of class 2 the answer is affirmative [**1**, Corollary]. However the answer is negative for groups of class 3. In fact, for each prime p there exists a nilpotent group of class 3 with cyclic center, which is residually K-linear but not residually finite, where K is a field containing the p^n-roots of unity, for all $n \geqslant 1$ [**1**, Theorem I(iii)]. It appears to us that K is exceedingly large and so the above question looks more interesting if we consider $\mathbf{Q}$-linearity. In this direction we prove the following:

PROPOSITION I. (i) *Let G be a residually $\mathbf{Q}$-linear nilpotent group with cyclic center. Then G is residually finite.*

(ii) *There exists a residually $\mathbf{Q}$-linear, but not residually finite, nilpotent group whose center is $\mathbf{Z} \oplus \mathbf{Z}$.*

Our notation follows that of [**1**]. However we need the following. Let G be a group. We say that G is a *C-group* if there exists a finite subset X of G such that $Z_1(G) = C_G(X)$. If $G/Z_i(G)$ is a C-group, for all $i \geqslant 0$, we say that G is a *C_0-group*. For example, any linear group, over a field, is a C_0-group. Then we can prove the following:

Received by the editors February 2, 1978.
AMS (MOS) subject classifications (1970). Primary 20D15, 20E25; Secondary 20H20.
Key words and phrases. Nilpotent group, residually finite, residually linear.

PROPOSITION II. *Let G be a nilpotent group such that $\langle x\rangle^G$ is finitely generated, for all $x \in G$. Then the following are equivalent*:

(i) *G is residually center-by-finite*,

(ii) *G is residually abelian-by-finite*,

(iii) *G is residually linear*,

(iv) *G is residually C*.

PROOF OF PROPOSITION I. (i) It follows from [3, Ex. 2.0] that $G \hookrightarrow \prod P_i \times R$, where P_i is a torsion-free $\mathbf{Q}$-linear group and R is a residually finite group. Denote the projections of G into P_i and R by π_i and π, respectively. Let z be a generator of $Z_1(G)$. Suppose that $\pi_i(z) \neq 1$, for some i. Since P_i is torsion-free we see that π_i is a monomorphism. Thus G is finitely generated and so residually finite. Suppose $\pi_i(z) = 1$, for all i. Then $G \hookrightarrow R$ and so G is residually finite.

(ii) Let p be a prime. Let H be a group generated by u, x_i, y_i, t_i, $i = 1, 2, \ldots,$ subject to the relations

$$[x_i, x_j] = [y_i, y_j] = [t_i, t_j] = 1, \quad \text{for all } i, j,$$

$$[x_i, y_i] = t_i^i, \quad [x_i, y_j] = [t_i, x_j] = [t_i, y_j] = 1 \quad \text{if } i \neq j, \text{ for all } i$$

$$[t_i, x_i] = [t_i, y_i] = u^{p^i}, \quad \text{for all } i; \quad u \text{ is central in } H.$$

H is a torsion-free nilpotent group of class 3, with $Z_1(G) = \langle u\rangle$. We will prove that H is residually finite. Set

$$H_m = \langle x_n, y_n, t_n, u^{p^m}, \quad \text{for } n > m\rangle.$$

Clearly $H_m \lhd G$, and $H_m \cap \langle u\rangle = \langle u^{p^m}\rangle$. Hence $\bigcap_{m\geq 1} H_m = \langle 1\rangle$. Since H/H_m is finitely generated, it follows that it is residually finite and that H is also. Let us consider the group G generated by u, z, x_i, y_i, t_i, $i = 1, 2, \ldots,$ subject to the relations

$$[x_i, x_j] = [y_i, y_j] = [t_i, t_j] = 1, \quad \text{for all } i, j,$$

$$[x_i, y_i] = t_i^i z, \quad [x_i, y_j] = [x_i, t_j] = [y_i, t_j] = 1 \quad \text{if } i \neq j, \text{ for all } i,$$

$$[t_i, x_i] = [t_i, y_i] = u^{p^i}, \quad \text{for all } i; \quad u, z \text{ are central in } G.$$

G is a nilpotent group of class 3 with $Z_1(G) = \langle u, z\rangle \cong \mathbf{Z} \oplus \mathbf{Z}$. We will prove that G is residually $\mathbf{Q}$-linear but not residually finite. Let $x \mapsto \bar{x}$ be a homomorphism of G into a finite group $\bar{G}$. Set $|\bar{G}| = m$. Since $\bar{G}$ is finite, there exist integers $r \neq s$ such that $\bar{x}_{rm} = \bar{x}_{sm}$. Then we get

$$\bar{1} = [\bar{x}_{rm}, \bar{y}_{sm}] = [\bar{x}_{sm}, \bar{y}_{sm}] = \bar{t}_{sm}^{sm}\bar{z} = \bar{z}.$$

Therefore $z \in R(G)$. Since $G/\langle z\rangle \cong H$, by the result of the preceding paragraph we obtain that $R(G) \subseteq \langle z\rangle$. Thus $R(G) = \langle z\rangle$. Put $F = \langle u, x_i, y_i,$ for all $i\rangle$. Clearly $F \lhd G$. Denote the isolator of F by $\sqrt{F}$. It is easily seen that $\sqrt{F} \cap R(G) = \langle 1\rangle$. It follows from the relations of G that $G/\sqrt{F} \cong \mathbf{Q}^+$. Therefore G is residually $\mathbf{Q}$-linear and the result follows.

In order to prove Proposition II we use the following results:

LEMMA 1. *Let G be a C_0-group. If H is a normal subgroup of G such that $H \cap Z_1(G)$ is finitely generated, then $H \cap Z_i(G)$ is finitely generated, for all $i \geqslant 1$.*

PROOF. It is an obvious modification of [**1**, Lemma 1].

LEMMA 2. *Let G be a C-group each of whose finitely generated subgroups satisfy the maximal condition on subgroups. Consider the following possibilities of G:*

(i) *$\Gamma_2(G)$ is finitely generated,*

(ii) *if $x \in G$, then $\langle x\rangle^G$ is finitely generated,*

(iii) *$Z_i(G)/Z_1(G)$ is finitely generated for all $i \geqslant 1$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If in addition G is nilpotent, then* (iii) $\Rightarrow$ (i).

PROOF. Since $\langle x\rangle^G \subseteq \langle x\rangle\Gamma_2(G)$, we see that (i) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (iii). First we observe that G is a C_0-group. Since homomorphic images of G satisfy (ii) it suffices to show that $G/Z_1(G)$ is a C-group. Let X be a finitely generated subgroup of G with $Z_1(G) = C_G(X)$. It follows from the hypothesis that $\langle X\rangle^G$ is finitely generated. Since $[X, G] \subseteq \langle X\rangle^G$ and the finitely generated subgroups of G satisfy the maximal condition on subgroups we see that $[X, G]$ is finitely generated. Suppose that $[x_1, g_1], \ldots, [x_n, g_n]$ are its generators, where $x_i \in X$ and $g_i \in G$. Set

$$H = \langle X, g_1, g_2, \ldots, g_n\rangle.$$

Clearly $[X, G] \subseteq [H, H]$. Set

$$N = \{a \in G \mid [a, H] \subseteq Z_1(G)\}.$$

The result follows if we prove that $Z_2(G) = N$. Trivially $Z_2(G) \subseteq N$. Moreover $[H, N, H] = [N, H, H] = \langle 1\rangle$, so P. Hall's lemma [**2**, Lemma 1.11] yields $[H, H, N] = \langle 1\rangle$ and hence $[X, G, N] = \langle 1\rangle$. Since $X \subseteq H$, we have $[N, X, G] = \langle 1\rangle$. Thus $[G, N, X] = \langle 1\rangle$, and $[G, N] \subseteq Z_1(G)$. This proves that $N \subseteq Z_2(G)$. The homomorphism $Z_2(G) \to \langle X\rangle^G \times \cdots \times \langle X\rangle^G$ in which

$$x \mapsto ([x, x_1], \ldots, [x, x_n]),$$

where $X = \langle x_1, x_2, \ldots, x_n\rangle$, proves that $Z_1(G/Z_1(G))$ is finitely generated. Since $G/Z_1(G)$ is a C_0-group, the result follows from Lemma 1. If we assume G nilpotent, then [**2**, Corollary 3.19] proves that (iii) $\Rightarrow$ (i).

LEMMA 3. *Let G be a nilpotent group with finite commutator and $G/Z_1(G)$ finitely generated. Then $G/Z_1(G)$ is finite.*

PROOF. Since $G/Z_1(G)$ is finitely generated, $Z_2(G)/Z_1(G)$ is isomorphic to a subgroup of a finite product $\prod\Gamma_2(G)$. Since finitely generated nilpotent groups with finite center are finite, the result follows.

PROOF OF PROPOSITION II. Trivially (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). (iv) $\Rightarrow$ (i). Clearly we may assume that G is a C-group. Lemma 2 yields that both $\Gamma_2(G)$ and $G/Z_1(G)$ are finitely generated. For any integer $n \geqslant 1$, denote $G/\Gamma_2(G)^n$ by G_n. Then $\Gamma_2(G_n)$ is finite and $G_n/Z_1(G_n)$ is finitely generated. It follows from Lemma 3 that G_n is center-by-finite. Let H_n be the complete inverse image of $Z_1(G_n)$ in G. H_n is of finite index in G and $[H_n, G] \subseteq \Gamma_2(G)^n$. Since $\Gamma_2(G)$ is finitely generated we have $\bigcap_{n \geqslant 1} \Gamma_2(G)^n = \langle 1 \rangle$. Therefore $\bigcap_{n \geqslant 1} [H_n, G] = \langle 1 \rangle$. The result follows.

REFERENCES

1. P. Menal, *Residual linearity for certain nilpotent groups*, Proc. Amer. Math. Soc. **68** (1978), 27–31.

2. R. B. Warfield, Jr., *Nilpotent groups*, Lecture Notes in Math., vol. 513, Springer-Verlag, Berlin and New York, 1976.

3. B. A. F. Wehrfritz, *Infinite linear groups*, Springer-Verlag, Berlin and New York, 1973.

SECCIÓ DE MATEMÀTIQUES UNIVERSITAT AUTÒNOMA DE BARCELONA, BARCELONA, SPAIN