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THE LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCES FOR SHEAVES WITH OPERATORS

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ABSTRACT. Two spectral sequences associated with a G-sheaf and a normal subgroup of G are given together with an application.

If H is a normal subgroup of a group G, the Lyndon-Hochschild-Serre spectral sequence relates the cohomology of H and of G/H to that of G. In this note we give two analogous spectral sequences in the cohomology theory of groups with coefficients in sheaves with operators. Applications include a direct proof of a theorem of Conner and Raymond [1].

1. Sheaves with operators. We start by briefly reviewing the cohomology of groups with coefficients in sheaves with operators. For details, we refer to [1], [2], and [3]. Let (G, X) be a continuous action of a group G on a topological space X. A G-sheaf over X is a sheaf $\mathbb{S}$ of abelian groups over X with an action $(G, \mathbb{S})$ such that the projection $\mathbb{S} \to X$ is equivariant and that for each $g \in G$ and $x \in X$, g: $\mathbb{S}_x \to \mathbb{S}_{gx}$ is a group homomorphism between the stalks. The categories of G-sheaves over X and of abelian groups are denoted, respectively, by $\mathcal{C}_X^G$ and $\mathcal{C}$. The functor Γ_X^G: $\mathcal{C}_X^G \to \mathcal{C}$ sends a G-sheaf $\mathbb{S}$ to the group $\Gamma(X, \mathbb{S})^G$ of G-invariant sections of $\mathbb{S}$ over X. For a G-sheaf $\mathbb{S}$ over X, the cohomology $H^\cdot(G, \mathbb{S})$ of G with coefficients in $\mathbb{S}$ is defined by $H^\cdot(G, \mathbb{S}) = R^\cdot\Gamma_X^G(\mathbb{S})$, where $R^\cdot\Gamma_X^G$ denotes the right derived functor of Γ_X^G.

2. The Lyndon-Hochschild-Serre spectral sequences for *G*-sheaves. Let H be a normal subgroup of G and let φ: $X \to X/H$ be the canonical projection onto the orbit space. The category $\mathcal{C}_X^G$ can be thought of as a subcategory of $\mathcal{C}_X^H$ in a natural manner. If M is a G-module, then the group $\Gamma^H(M) = M^H$ of H-invariant elements of H becomes naturally a G/H-module. Thus the functor Γ_X^H: $\mathcal{C}_X^H \to \mathcal{C}$ sends $\mathcal{C}_X^G$ into the category $\mathcal{C}^{G/H}$ of G/H-modules, since it is factorized as $\Gamma_X^H = \Gamma^H \circ \Gamma_X$. We denote by $\mathcal{C}_{X/H}$ the category of sheaves of abelian groups over X/H. The functor φ_*^H: $\mathcal{C}_X^H \to \mathcal{C}_{X/H}$ assigns, by definition, to each H-sheaf $\mathbb{S}$ the sheaf over X/H determined by the presheaf $U \to \Gamma(\varphi^{-1}(U), \mathbb{S})^H$. The restriction of φ_*^H to the subcategory $\mathcal{C}_X^G$

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can be thought of as a functor into the category $\mathcal{C}_{X/H}^{G/H}$ of G/H-sheaves over X/H in a natural manner.

LEMMA. *If $\mathcal{I}$ is an injective G-sheaf over X, then*
(1) $\mathcal{I}$ is an injective H-sheaf over X,
(2) $\Gamma_X^H(\mathcal{I})$ is an injective G/H-module,
(3) $\varphi_*^H(\mathcal{I})$ is a $\Gamma_{X/H}^{G/H}$-acyclic G/H-sheaf over X/H, i.e.

$$R^n\Gamma_{X/H}^{G/H}\big(\varphi_*^H(\mathcal{I})\big)\ \Big(= H^n\big(G/H, \varphi_*^H(\mathcal{I})\big)\Big) = 0, \quad \textit{for } n \geqslant 1.$$

PROOF. (1) Set $O = \mathbf{Z}(H)$ in [2, Lemma 5.6.2]. (2) Recall the factorization $\Gamma_X^H = \Gamma^H \circ \Gamma_X$. By [2, Corollaire de Proposition 5.1.3], Γ_X sends an injective G-sheaf to an injective G-module. On the other hand, Γ^H sends an injective G-module to an injective G/H-module. (3) By (1) above and [2, Corollaire de Proposition 5.1.3], the sheaf $\varphi_*^H(\mathcal{I})$ is a flabby sheaf over X/H. Consider the spectral sequence [3, (1.2)] for $\mathcal{S} = \varphi_*^H(\mathcal{I})$:

$$E_2^{p,q} = H^p\Big(G/H, H^q\big(X/H, \varphi_*^H(\mathcal{I})\big)\Big) \Rightarrow H^n\big(G/H, \varphi_*^H(\mathcal{I})\big).$$

Since $\varphi_*^H(\mathcal{I})$ is flabby, the spectral sequence degenerates to yield

$$\begin{aligned} H^n\big(G/H, \varphi_*^H(\mathcal{S})\big) &\simeq H^n\Big(G/H, H^0\big(X/H, \varphi_*^H(\mathcal{I})\big)\Big) \\ &= H^n\big(G/H, H^0(X, \mathcal{I})^H\big). \end{aligned}$$

Also consider the Lyndon-Hochschild-Serre spectral sequence for the G-module $H^0(X, \mathcal{I})$:

$$E_2^{p,q} = H^p(G/H, H^q(H, H^0(X, \mathcal{I}))) \Rightarrow H^n(G, H^0(X, \mathcal{I})).$$

By (1) above and [2, Corollaire de Proposition 5.1.3], $H^0(X, \mathcal{I}) = \Gamma_X(\mathcal{I})$ is injective either as G-module or as H-module. Hence $H^n(H, H^0(X, \mathcal{I})) = 0 = H^n(G, H^0(X, \mathcal{I}))$ for $n \geqslant 1$. Therefore we have $H^n(G/H, H^0(X, \mathcal{I})^H) = H^n(G/H, H^0(H, H^0(X, \mathcal{I}))) = H^n(G, H^0(X, \mathcal{I})) = 0$, for $n \geqslant 1$. Q.E.D.

THEOREM. *If $\mathcal{S}$ is a G-sheaf over X and if H is a normal subgroup of G, then for each nonnegative integer q, the group $H^q(H, \mathcal{S})$ has a canonical structure of G/H-module and the sheaf $R^q\varphi_*^H(\mathcal{S})$ that of G/H-sheaf over X/H and there are two spectral sequences*

$$'E_2^{p,q} = H^p\big(G/H, R^q\varphi_*^H(\mathcal{S})\big) \Rightarrow H^n(G, \mathcal{S}),$$

$$''E_2^{p,q} = H^p(G/H, H^q(H, \mathcal{S})) \Rightarrow H^n(G, \mathcal{S}).$$

PROOF. Lemma (1) shows that $R^{\cdot}\Gamma_X^H(\mathcal{S})$ and $R^{\cdot}\varphi_*^H(\mathcal{S})$ can be computed by taking a G-injective resolution of $\mathcal{S}$. Thus, for each q, $H^q(H, \mathcal{S}) = R^q\Gamma_X^H(\mathcal{S})$ has a canonical structure of G/H-module and $R^q\varphi_*^H(\mathcal{S})$ that of G/H-sheaf over X/H. With the aid of the lemma, [2, Théorème 2.4.1] is applied to the commutative diagram

$$\begin{tikzcd}
& \mathcal{C}_X^G \arrow[dl, "\Gamma_X^H"'] \arrow[dr, "\varphi_*^H"] \arrow[dd, "\Gamma_X^G"] & \\
\mathcal{C}^{G/H} \arrow[dr, "\Gamma^{G/H}"'] & & \mathcal{C}_{X/H}^{G/H} \arrow[dl, "\Gamma_{X/H}^{G/H}"] \\
& \mathcal{C} &
\end{tikzcd}$$

to obtain the spectral sequences.

3. An application. (3.10) Theorem in [1] is obtained from the above theorem as follows. Let (N, W) be a properly discontinuous action of a group N on W. A representation $\Phi: N \to GL(2k, \mathbf{Z})$ defines an N-sheaf structure on the constant sheaf $\mathcal{Z}^{2k} = W \times \mathbf{Z}^{2k}$. Let $L \subset N$ be a normal subgroup satisfying

(i) L acts freely on W,

(ii) $L \subset \operatorname{Ker} \Phi$.

Denote by $\mu: W \to W/L = B$ the canonical projection onto the orbit space. By (ii), Φ defines an N/L-sheaf structure on the constant sheaf $\mathcal{Z}^{2k} = B \times \mathbf{Z}^{2k}$ on B. We set $X = W$, $G = N$, $H = L$, $\varphi = \mu$ and $\mathcal{S} = \mathcal{Z}^{2k}$ in the first spectral sequence of the theorem. The condition (i) implies that $R^q\mu_*^L(\mathcal{Z}^{2k}) = 0$, for $q > 1$. Hence the spectral sequence degenerates to yield $H^n(N/L, \mu_*^L(\mathcal{Z}^{2k})) \simeq H^n(N, \mathcal{Z}^{2k})$. Moreover, from conditions (i) and (ii), we see that the sheaf $\mu_*^L(\mathcal{Z}^{2k}) \to B$ is identical with $\mathcal{Z}^{2k} \to B$. Hence we get

$$H^n(N/L, \mathcal{Z}^{2k}) \simeq H^n(N, \mathcal{Z}^{2k}).$$

For another application, see the proof of [3, Corollary 4.3].

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