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ONE-SIDED CLUSTER-SET THEORY IN A POLYDISC

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ABSTRACT. In this paper, we develop a technique which reduces the investigation of the boundary behavior of a *single* function f in $H^\infty(U^n)$, $n > 1$, to the study of the cluster sets of a *sequence of functions* in $H^\infty(U)$. We also demonstrate the significance and usefulness of one-sided cluster-set theory.

1. Introduction. The purpose of this paper is threefold. First, we develop a technique which reduces the investigation of the boundary behavior of a *single function* f in $H^\infty(U^n)$, $n > 1$, to the study of the cluster sets of a *sequence of functions* in $H^\infty(U)$. This technique was inspired to a large extent by the ingenious methods and ideas which were first introduced in the one-variable theory of cluster sets by Doob [**5**], [**6**]. (In particular, see Doob [**5**, p. 441, Theorem 4.2].) At the outset, we impose a limitation on the possible approaches to a boundary point P on T^n. Thus, at least initially, we restrict ourselves to only one-sided cluster sets. (For the definition, see §2.) One advantage of this restricted approach to a boundary point P on T^n is that several fundamental results of the one-variable theory of cluster sets become tractable in the several-variable situation. For example, in Theorem 3 we are able to extend the one-sided Iversen theorem to the polydisc U^n. In addition to the one-sided approach, another ingredient of our technique is contained in the statements and in the proofs of Lemmas 4 and 5. Second, we use our technique to prove a strong form of the Gross Cluster-Value Theorem [**8**] in the polydisc U^n (Theorem 6). Third, we provide some applications of our results. For instance, in Corollary 9 we extend and generalize Lindelöf's important classical theorem on the asymptotic values of analytic functions [**10**]. We end the paper with an open question concerning the Gehring-Lohwater phenomenon [7] in the polydisc U^n.

For the sake of simplicity, we present our theorems in the unit polydisc U^2. We remark, however, that our results remain valid for a polydisc in $\mathbf{C}^n$, where $n \geqslant 2$.

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2. Definitions and terminology. The topological concepts we use in this paper are relative to the extended complex plane $\mathbf{C}_\infty$. We denote the complement of a set A by $\tilde{A}$.

Let the function $\zeta = f(z, w)$ map U^2 into $\mathbf{C}_\infty$. Let γ_1 and γ_2 be two paths in U which terminate at $e^{i\theta_1}$ and $e^{i\theta_2}$, respectively. (We use the terms "path" and "arc" interchangeably.) If $\gamma = \gamma_1 \times \gamma_2$, then we call the cluster set $C_\gamma(f, (e^{i\theta_1}, e^{i\theta_2}))$ the *product path cluster set of f at* $(e^{i\theta_1}, e^{i\theta_2})$. This cluster set is defined as the set of all values $\zeta \in \mathbf{C}_\infty$ such that there is a sequence $\{(z_n, w_n)\}$ with the properties that $(z_n, w_n) \in \gamma$ for each n, $(z_n, w_n) \to (e^{i\theta_1}, e^{i\theta_2})$, and $f(z_n, w_n) \to \zeta$. If both γ_1 and γ_2 lie in a Stolz angle, then we call γ a (*product*) *Stolz path*. If both γ_1 and γ_2 are tangential paths, then we term γ a (*product*) *tangential path*. We define the *principal cluster set of f at* $P = (e^{i\theta_1}, e^{i\theta_2})$ as the intersection of all the cluster sets $C_\gamma(f, P)$, where the intersection is taken over all product paths γ terminating at P.

Let Δ_1 and Δ_2 be two Stolz angles in U which terminate at $e^{i\theta_1}$ and $e^{i\theta_2}$, respectively, and let $\Delta = \Delta_1 \times \Delta_2$ denote the product Stolz angle which terminates at $P = (e^{i\theta_1}, e^{i\theta_2})$. Then the cluster set $C_\Delta(f, P)$ is defined as the set of values $\zeta \in \mathbf{C}_\infty$ such that there is a sequence $\{(z_n, w_n)\}$ with the properties that $(z_n, w_n) \in \Delta$ for each n, $(z_n, w_n) \to (e^{i\theta_1}, e^{i\theta_2})$, and $f(z_n, w_n) \to \zeta$. We define $C_A(f, P)$, the *outer angular cluster set of f at P*, as the set

$$C_A(f, p) = \bigcup C_\Delta(f, P),$$

where the union is taken over all (product) Stolz angles Δ terminating at P. A value ζ in $C_A(f, P)$ is called an (*outer*) *angular cluster value of f at P*.

In the sequel, we also require the following definitions and terminology associated with certain one-sided approaches to a point P on T^2. If $0 < \eta \leqslant \pi$, we call

$$\Sigma_{SS}(\eta) = \{(z, w) | 3\pi/2 - \arg(z - 1) \leqslant \eta, 3\pi/2 - \arg(w - 1) \leqslant \eta\}$$

the *η-sector from the south-south*. Henceforth, we tacitly assume without loss of generality that the point P on T^2 under consideration is the point (1, 1). Let $\bar{C}_{SS}(\eta)$ denote the cluster set of $f \in H^\infty(U^2)$ at (1, 1) in the sector $\Sigma_{SS}(\eta)$, let

$$C_{SS}(\eta) = \bigcup_{\eta' < \eta} \bar{C}_{SS}(\eta'),$$

and let

$$C_{SS}(0) = \bigcap_{\eta > 0} C_{SS}(\eta) = \bigcap_{\eta > 0} \bar{C}_{SS}(\eta).$$

Thus, $C_{SS}(0)$ is the set of *tangential cluster values from the south-south of f at* (1, 1).

If $\eta > 0$, we denote by $R_{SS}(\eta)$ the set of values assumed by f arbitrarily near (1, 1) in some η'-sector (from the south-south) with $\eta' < \eta$, and we define $R_{SS}(0) = \bigcap_{\eta > 0} R_{SS}(\eta)$. Finally, we define $\hat{B}_{SS}$, the *distinguished boundary cluster set of f at* (1, 1) *from the south-south*, as the intersection

($n \geqslant 1$) of the closure of the set of cluster values of f at points $(e^{i\theta}, e^{i\phi})$ on T^2 such that $-1/n < \theta, \phi < 0$. The sets Σ_{SN}, Σ_{NN}, etc., and the corresponding one-sided cluster sets, are defined analogously. (See **[3]** for a detailed discussion of some of the cluster sets defined above.)

3. The technique and the results with applications. We begin this section with some preliminary results of Doob **[6]** which, thanks to the one-sided approach, readily extend to the polydisc U^2.

LEMMA 1. *Let $f \in H^\infty(U^2)$. If $\alpha \in C_{SS}(0)$, then $|\alpha| \leqslant \sup|\beta|$, where the supremum is taken over all points β in $\hat{B}_{SS}$.*

THEOREM 2. *Let $f \in H^\infty(U^2)$. Then the set $C_{SS}(0) \cap \tilde{\hat{B}}_{SS}$ is open.*

Since the proofs of Lemma 1 and Theorem 2 involve only the iterations of certain one-variable ideas, we omit the proofs here. (For the details of the proofs in the one-variable case, we refer the reader to Doob **[6]**.) We remark that *mutatis mutandis* analogous results hold for the other directions of approach. Thus, for example, Theorem 2 remains valid if $C_{SS}(0)$ is replaced by, say, $C_{SN}(0)$ and $\hat{B}_{SS}$ is replaced by the corresponding set $\hat{B}_{SN}$. Similar remarks apply to our subsequent results which we present for only one particular direction of approach. It is this property which is the *raison d'être* for our use of one-sided cluster sets.

The next result, Theorem 3, which we need in the sequel, is a generalization to the polydisc U^2 of Doob's extension of the Iversen Theorem **[9]**. This result contains the classical Iversen Theorem (see Doob **[6**, p. 463], and Collingwood and Lohwater **[4**, p. 89] and **[2]**). In addition to a standard normal family argument, the proof of Theorem 3 rests on a remarkable geometric property which is enjoyed by certain Möbius transformations. In the argument which follows, we briefly describe this property.

We suppose, for the sake of argument, that there is a point α in $C_{SS}(\eta) \cap \tilde{\hat{B}}_{SS}$ which is also a boundary point of $C_{SS}(\eta)$, where $\eta > 0$. Then there is a sequence $\{(z_n, w_n)\}$ in the η'-sector $\Sigma_{SS}(\eta')$, for some $\eta' < \eta$, such that $(z_n, w_n) \to (1, 1)$ and $f(z_n, w_n) \to \alpha$. Let K_n and L_n denote the Möbius transformations which map U onto U and satisfy the conditions $K_n(1) = L_n(1) = 1$, $K_n(0) = z_n$, and $L_n(0) = w_n$. The properties of K_n and L_n now imply that there is a closed polydisc $\bar{D}$ about the origin such that for all n, the mappings $(z, w) \to (K_n(z), L_n(w))$ take $\bar{D}$ into the interior of the sector $\Sigma_{SS}(\eta)$. Now we define $f_n(z, w) = f(K_n(z), L_n(w))$, and we observe that since $f \in H^\infty(U^2)$, $\{f_n\}$ is a normal family. The rest of the argument is *mutatis mutandis* the same as in Doob **[6**, p. 467]. We obtain that α is in $C_{SS}(0)$. Since $C_{SS}(0) \cap \tilde{\hat{B}}_{SS}$ is open by Theorem 2, α cannot be a boundary point of $C_{SS}(\eta)$. This contradiction completes the proof of Theorem 3.

THEOREM 3 (THE DOOB-IVERSEN THEOREM). *Let $f \in H^\infty(U^2)$. If $\eta > 0$, then the set $C_{SS}(\eta) \cap \tilde{\hat{B}}_{SS}$ is open.*

In the following key lemma, whose one-variable version is due to Doob **[6**, p. 463], we exhibit by a concrete construction the second ingredient of our technique. That is, in the proof of the key lemma we demonstrate the manner in which a *sequence of functions* in $H^\infty(U)$ can be constructed to reflect some of the desired properties of a *single function* in $H^\infty(U^2)$.

LEMMA 4 (KEY LEMMA). *Let* $f \in H^\infty(U^2)$, *and let* $\alpha \in C_{SN}(0) \cap \tilde{\hat{B}}_{SN}$. *Let* $D(\alpha, \hat{B}_{SN})$ *denote the open component of* $\mathbf{C}_\infty - \hat{B}_{SN}$ *containing* α. *Then, aside from at most one exceptional point*, $R_{SN}(0) \supseteq D(\alpha, \hat{B}_{SN})$.

PROOF. By Theorem 2, the set $C_{SN}(0) \cap \tilde{\hat{B}}_{SN}$ is open. Hence, it follows that $D(\alpha, \hat{B}_{SN}) \subseteq C_{SN}(0)$. Let $\{(z_n, w_n)\}$ be a sequence of points in U^2 such that $\{(z_n, w_n)\}$ converges tangentially from the south-north to $(1, 1)$, and such that $f(z_n, w_n) \to \alpha$ as $n \to \infty$. For each fixed n, we choose an open arc $A_n = \widehat{a_n, 1}$ of T which abuts $z = 1$ from the south, and we choose an open arc $B_n = \widehat{1, b_n}$ of T which abuts $w = 1$ from the north. Moreover, we require that the lengths $\|A_n\|$, $\|B_n\|$ of the arcs A_n and B_n, respectively, tend to zero, but so slowly that $\mu(z_n, A_n)$, $\mu(w_n, B_n) \to 1$, where $\mu(z_n, A_n)$ denotes the harmonic measure of A_n at z_n with respect to the unit disc U.

Next, for each n, we denote by ϕ_n the Möbius transformation which maps U onto U and satisfies the conditions $\phi_n(a_n) = 1$ and $\phi_n(z_n) = w_n$. If $\|\phi_n(A_n)\| \leqslant \|B_n\|$ for infinitely many n, then we set $f_n(z) = f(z, \phi_n(z))$, and we consider the function-arc sequence $\{f_n, A_n\}$. (Otherwise, we let $c_n = \phi_n^{-1}(b_n)$, and we consider the function-arc sequence $\{g_n, B_n\}$, where $g_n(w) = f(\phi_n^{-1}(w), w)$.) We now proceed to show that this is the desired reduction to the one-variable situation.

Let S denote the boundary cluster set of the sequence $\{f_n\}$ relative to the sequence of arcs $\{A_n\}$. (For the definition of this cluster set, see Doob **[6**, p. 466].) Since $S \subseteq \hat{B}_{SN}$ and $\alpha \notin \hat{B}_{SN}$ by assumption, it is obvious that $\alpha \notin S$. Now the above construction implies that $f_n(z_n) = f(z_n, w_n) \to \alpha$ and $\mu(z_n, A_n) \to 1$. Thus, it follows from a one-variable result of Doob (see **[6**, p. 466, Theorem 5.1] or **[5**, p. 441, Theorem 4.2]) that at most one point of $D(\alpha, S) \supseteq D(\alpha, \hat{B}_{SN})$ is omitted by the sequence $\{f_n\}$. Hence, aside from at most one point of $D(\alpha, \hat{B}_{SN})$, f assumes every point of $D(\alpha, \hat{B}_{SN})$. Finally, we observe that this conclusion remains valid if the domain of f is replaced by the intersection of any neighborhood of $(1, 1)$ with the interior of any η-sector $(\eta > 0)$ from the south-north. This completes the proof of the key lemma.

LEMMA 5. *Let* $f \in H^\infty(U^2)$. *If* $\alpha \in C_{SS}(\eta) \cap \tilde{\hat{B}}_{SS}$, $\eta > 0$, *and if* $C_{SS}(0) \cap D(\alpha, \hat{B}_{SS}) = \varnothing$, *then*

$$[C_{SS}(\eta) - R_{SS}(\eta)] \cap D\left(\alpha, \tilde{\hat{B}}_{SS}\right) = \varnothing.$$

Since the proof of Lemma 5 is similar to the proof of Theorem 3, we omit it.

We are now in a position to apply the above results and prove the Gross Cluster-Value Theorem for functions in $H^\infty(U^2)$.

THEOREM 6 (GROSS CLUSTER-VALUE THEOREM). *Let* $f \in H^\infty(U^2)$. *Then every angular cluster value of* f *at* $(1, 1)$ *which is a limit point of* $\tilde{R}(f, (1, 1))$ *is a principal value of* f *at* $(1, 1)$.

PROOF. Suppose the conclusion of the theorem is false. Then there is an angular sequence $\{(z_n, w_n)\}$ in U^2 with limit $(1, 1)$ along which f has limit α, where α is a limit point of $\tilde{R}(f, (1, 1))$, and there is a product path $\gamma = \gamma_1 \times \gamma_2$ terminating at $(1, 1)$ such that $\alpha \notin C_\gamma(f, (1, 1))$. Since in every locally compact, connected, and locally connected space any two points are the endpoints of a simple path in the space (see, for example, Whyburn [**12**, p. 36]), we may assume that γ_1 and γ_2 are simple. We further assume that γ_j $(j = 1, 2)$ has its initial point, different from 1, on T. Thus γ divides U^2 into four product domains. By taking a suitable subsequence of the sequence $\{(z_n, w_n)\}$, if necessary, we may assume that the sequence lies in one of these domains, say $D = D_1 \times D_2$. Finally, we assume without loss of generality that the portion of the boundary of D_j which lies on T abuts 1 from above. Now let ϕ_j map D_j conformally onto U such that $\phi_j(1) = 1$ $(j = 1, 2)$. Then it follows from a lemma of Doob [**6**, p. 464] that the sequence $\{(\phi_1(z_n), \phi_2(w_n))\}$ lies in some η'-sector from the south-south, where $\eta' < \pi$. Thus, if we let $g(z, w) = f(\phi_1(z), \phi_2(w))$, $(z, w) \in D$, and choose $\eta > \eta'$, then $\alpha \in C_{SS}(g, \eta) \cap \hat{B}_{SS}(g)$ and α is a limit point of $\tilde{R}_{SS}(g, \eta)$.

In order to obtain a contradiction, we next examine two mutually exclusive cases. First, if $\alpha \in C_{SS}(g, 0) \cap \hat{B}_{SS}(g)$, then by Lemma 4 the set $R_{SS}(g, 0)$, and *a fortiori* the set $R_{SS}(g, \eta)$ contains at least a deleted neighborhood of α. Second, if $\alpha \notin C_{SS}(g, 0) \cap \hat{B}_{SS}(g)$, then an easy argument shows that $C_{SS}(g, 0) \cap D(\alpha, \hat{B}_{SS}(g)) = \varnothing$. Hence, by Lemma 5, $R_{SS}(g, \eta)$ contains at least a deleted neighborhood of α. Since α was a limit point of $\tilde{R}_{SS}(g, \eta)$, we have arrived at a contradiction. Thus, the proof of the Gross Cluster-Value Theorem is complete.

Our first corollary is an immediate consequence of the Gross Cluster-Value Theorem.

COROLLARY 7. *Let* $f \in H^\infty(U^2)$. *If* $\sigma = \sigma_1 \times \sigma_2$ *is a product Stolz path in* U^2 *terminating at* $(1, 1)$, *then*

$$C_\sigma(f, (1, 1)) \cap \partial C(f, (1, 1)) \subseteq C_\gamma(f, (1, 1)),$$

where $\gamma = \gamma_1 \times \gamma_2$ *is any product path in* U^2 *terminating at* $(1, 1)$, *and where* $\partial C(f, (1, 1))$ *denotes the boundary of the cluster set of* f *at* $(1, 1)$. *Moreover, if* $C(f, (1, 1))$ *is nowhere dense, then the outer angular cluster set of* f *at* $(1, 1)$ *is equal to the principal cluster set of* f *at* $(1, 1)$.

In order to underscore the scope of the foregoing results, we next present a slight extension of Lindelöf's classical theorem on the asymptotic values of analytic functions [**10**].

THEOREM 8. *Let $f \in H^\infty(U^2)$, and suppose that $\|f\|_\infty \leq 1$. Let $\gamma = \gamma_1 \times \gamma_2$ be a product path in U^2 terminating at* (1, 1), *and let $\Delta = \Delta_1 \times \Delta_2$ be a product Stolz angle with vertex at* (1, 1). *Then there is a constant $\alpha = \alpha(\Delta)$, $0 < \alpha \leq 1$, such that*

$$\overline{\lim_{\substack{(z,w)\to(1,1)\\(z,w)\in\Delta}}} |f(z, w)| \leq \left[\overline{\lim_{\substack{(z,w)\to(1,1)\\(z,w)\in\gamma}}} |f(z, w)| \right]^\alpha .$$

Given our earlier results, the proof of Theorem 8 involves only minor modifications of a technique of Cameron and Storvick [**1**], and so we omit the proof.

If in addition to the hypotheses of Theorem 8 we assume that the cluster set $C(f, (1, 1))$ of f at (1, 1) is nowhere dense, then we may take α to be 1. This observation is a novelty, which we feel is worth mentioning even in the one-variable case. It is an immediate consequence of Corollary 7.

COROLLARY 9. *Let $f \in H^\infty(U^2)$, and suppose that $\|f\|_\infty \leq 1$. If $C(f, (1, 1))$ is nowhere dense, then $\overline{\lim}_\Delta |f(z, w)| \leq \overline{\lim}_\gamma |f(z, w)|$, where Δ and γ have the same meaning as in Theorem* 8.

We conclude this paper with an open question related to what we termed in the Introduction as the "Gehring-Lohwater phenomenon." In the one-variable case, Gehring and Lohwater [7] proved the following beautiful theorem.

THEOREM 10 (THE GEHRING-LOHWATER THEOREM). *Let $f = u + iv \in H^\infty(U)$, and let α and β be any two paths in U terminating at* 1. *If $u \to a$ and $v \to b$ as $z \to 1$ along α and β respectively, then $f \to a + ib$ uniformly as $z \to 1$ in any Stolz angle.*

The question is whether or not this generalization of Lindelöf's theorem remains valid in the polydisc U^2.

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