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UNBOUNDED UNIFORMLY ABSOLUTELY CONTINUOUS SETS OF MEASURES

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ABSTRACT. It is shown that a uniformly absolutely continuous set of finitely additive measures can be decomposed into bounded and finite dimensional parts.

1. Introduction. It is well known that a uniformly absolutely continuous set G of (finitely additive) measures need not be bounded. One can, for example, let δ be a finite sum of atoms (= two-valued measures) and G be the set A_δ of all δ-continuous measures. We will show that this is "the only way in which G can be unbounded", in that G can be decomposed into bounded and finite dimensional parts. Consequences include a boundedness criterion for G in terms of atoms as well as the equivalence of the pointwise boundedness and boundedness of G.

Suppose S is a set, F is a field and Σ is a σ-field of subsets of S, $\mathrm{ba}(F)$ ($\mathrm{ca}(\Sigma)$) is the set of bounded and additive (countably additive) functions from F into R (= reals). For $G \subseteq \mathrm{ba}(F)$ we will denote by G^+ the set of nonnegatively valued elements of G. For $\lambda \in \mathrm{ba}(F)^+$ and $\eta \in \mathrm{ba}(F)$ we will denote the λ-continuous part of η by $P_\lambda(\eta)$, the total variation function of η by $|\eta|$ and the set of λ-continuous elements of $\mathrm{ba}(F)$ by A_λ. For a discussion of the lattice operations see **[4]**.

2. The atomic-nonatomic decomposition of Sobczyk and Hammer. An *atom* (of $\mathrm{ba}(F)$) is an element of $\mathrm{ba}(F)$ whose range contains exactly two elements. We will denote the set of all atoms whose nonzero value is 1 by T. If $\mu \in \mathrm{ba}(F)^+$, then E in F is a μ-*atom* if the contraction of μ to E is an atom; that is, if for each $V \in F$ we have either $\mu(E \cap V)$ or $\mu(E - V)$ is zero. An element η of $\mathrm{ba}(F)$ is *atomic* (= discrete in **[6]**) if η is zero or a sum of atoms and *nonatomic* if there is no atom in $\mathrm{ba}(F)^+$ less than or equal to $|\eta|$. In $\mathrm{ca}(\Sigma)$ this definition of atomic is equivalent to the statement that each μ-positive set contains a μ-atom and is more suitable (though not equivalent) in $\mathrm{ba}(F)$ where definitions which are dependent on sets of measure zero do not carry over well as with the notions of mutual singularity and absolute continuity.

A *subdivision* of $E \in F$ is a finite disjoint subset of F whose union is E. A

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refinement of a subdivision D of E is a subdivision H of E which is such that each V in D is the union of the set $H(V) = \{I \in H | I \subseteq V\}$.

As noted in **[6]** a pair of atoms is either mutually singular or linearly dependent. If δ is an atom and $\eta \in \mathrm{ba}(F)$ with $|\eta| \wedge |\delta| = 0$ we have a slight improvement of the ε-Hahn decomposition (of any pair of mutually singular elements of $\mathrm{ba}(F)$) in **[2]** in that we can select for $\varepsilon > 0$ an $E \in F$ such that $|\eta|(S) < |\eta|(E) + \varepsilon$ and $|\delta|(E) = 0$. If η is also an atom, then we can require $|\eta|(S) = |\eta|(E)$. Therefore by induction we have, for any disjoint (= pairwise mutually singular) sequence $(\mu_i)_{i=1}^M$ of atoms there exists a subdivision $D = \{E_i | i = 1,2, \ldots, M\}$ such that $\mu_i(E_i) = \mu_i(S)$ for each $i \leqslant M$. (Note that "disjoint" is used here in the sense of the lattice, $\mathrm{ba}(F)$, and not in the sense of the introduction in **[6]** although, as noted there, for a finite sequence of atoms the two notions are equivalent **[6**, p. 843**]**.) Consequently we have:

2.1. LEMMA. *If H is a subdivision of S and $(\mu_i)_{i=1}^M$ is a disjoint sequence of atoms, then there exists a refinement $D = \{E_i | i = 1, 2, \ldots, K\}$ of H such that $\mu_i(E_i) = \mu_i(S)$ for each $i \leqslant M$.*

The following theorem is due to Sobczyk and Hammer **[6]**.

2.2. THEOREM. *Each μ in $\mathrm{ba}(F)^+$ admits a decomposition $\mu = \mu_0 + \mu'$ such that*:

(1) *the measures μ_0 and μ' are mutually singular elements of* $\mathrm{ba}(F)^+$;

(2) *the measure μ' is the sum of a disjoint sequence $(\mu_i)_{i=1}^\infty$ where each μ_i is either zero or an atom of* $\mathrm{ba}(F)^+$;

(3) *for each $\varepsilon > 0$ there exists a subdivision D of S such that $\mu_0(E) < \varepsilon$ for each $E \in D$.*

We can also obtain a separation of μ in $\mathrm{ba}(F)^+$ into atomic and nonatomic parts via the Riesz decomposition theorem as in **[5**, p.143**]**. In particular the set of atomic elements is the smallest band containing T and the set of nonatomic elements is the complementary band, that is the band $T^\perp = \{\eta \in \mathrm{ba}(F) | \, |\eta| \wedge t = 0 \text{ for each } t \in T\}$.

That these two decompositions are the same follows from:

2.3. LEMMA. *If $\eta \in \mathrm{ba}(F)$, then the following two statements are equivalent*:

(1) *if $t \in T$, then $|\eta| \wedge t = 0$;*

(2) *for each $\varepsilon > 0$ there exists a subdivision D of S such that $|\eta|(E) < \varepsilon$ for each $E \in D$.*

The proof of this is essentially a reproduction of arguments given in **[6**, Lemmas 4.1 and 4.2**]** and is hence omitted.

We conclude this section by noting that the representation $P_\lambda(\mu) = \sup_k \mu \wedge k\lambda$ for μ, λ in $\mathrm{ba}(F)^+$ (see, for example, **[1]**) easily implies:

2.4. LEMMA. *If each of* λ *and* δ *is in* $\mathrm{ba}(F)^+$ *and* $\lambda \wedge \delta = 0$, *then*
(1) $P_{\lambda+\delta}(\eta) = P_\lambda(\eta) + P_\delta(\eta)$ *for each* $\eta \in \mathrm{ba}(F)$,
(2) $P_\lambda(\mu) \wedge P_\delta(\mu) = 0$ *for each* $\mu \in \mathrm{ba}(F)^+$.

3. The Decomposition. The proof of the main theorem will involve the following finitely additive version of a theorem of Saks [**3**, p. 308].

3.1. LEMMA. *Suppose* $\mu \in \mathrm{ba}(F)^+$, $\varepsilon > 0$ *and* $(\mu_i)_{i=0}^\infty$ *is as in* 2.2. *Then there exists a positive integer* M *and a subdivision* $D = \{E_i | i = 1,2, \ldots, K\}$ *of* S *such that* $\lambda(E_i) < \varepsilon$ *for each* $i \leqslant K$ *where* $\lambda = \mu_0 + \Sigma_{i=M+1}^\infty \mu_i$.

PROOF. Let M be such that $\Sigma_{i=M+1}^\infty \mu_i(S) < \varepsilon/2$ and D be a subdivision of S such that if $E \in D$, then $\mu_0(E) < \varepsilon/2$. Then for each $E \in D$ we have

$$\lambda(E) = \mu_0(E) + \sum_{i=M+1}^{\infty} \mu_i(E) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For $\mu \in \mathrm{ca}(\Sigma)^+$ we can of course obtain for each $i \leqslant M$ a μ-atom V_i such that $\mu(V_i) = \mu_i(V_i) = \mu_i(S)$ and therefore we have:

3.2. COROLLARY (SAKS). *If* $\mu \in \mathrm{ca}(\Sigma)^+$ *and* $\varepsilon > 0$, *then there exists a subdivision* D *of* S *such that for each* $E \in D$ *either* $\mu(E) < \varepsilon$ *or* E *is a* μ*-atom.*

PROOF. Let M, K, λ and $(E_i)_{i=1}^K$ be as in 3.1. For each $i \leqslant M$ we have $\mu_i \wedge (\mu - \mu_i) = 0$ hence there exists a V_i such that $\mu(V_i) = \mu_i(V_i) = \mu_i(S)$ and $(\mu - \mu_i)(V_i) = 0$; hence V_i is a μ-atom. Let $V = \bigcup_{i=1}^M V_i$, then

$$D = \{V_i | i \leqslant M\} \cup \{E_i \sim V | i \leqslant K\}$$

is the desired subdivision since for each $i \leqslant K$ we have

$$\begin{aligned}\mu(E_i \sim V) &= \mu_0(E_i \sim V) + \sum_{j=1}^{M} \mu_j(E_i \sim V) + \sum_{j=M+1}^{\infty} \mu_j(E_i \sim V)\\ &\leqslant \mu_0(E_i) + \sum_{j=M+1}^{\infty} \mu_j(E_i) = \lambda(E_i) < \varepsilon.\end{aligned}$$

3.3. THEOREM. *Suppose* $G \subseteq \mathrm{ba}(F)$ *is uniformly absolutely continuous with respect to* $\mu \in \mathrm{ba}(F)^+$. *Then there exist two subsets* G_1 *and* G_2 *of* $\mathrm{ba}(F)$ *such that*:
(1) $G \subseteq G_1 + G_2$;
(2) *each of* G_1 *and* G_2 *is uniformly absolutely continuous with respect to* μ;
(3) *if* $\eta_1 \in G_1$, *and* $\eta_2 \in G_2$, *then* $|\eta_1| \wedge |\eta_2| = 0$;
(4) $G_1 \subseteq A_\delta$ *where* δ *is a finite sum of atoms, and*
(5) G_2 *is bounded.*

PROOF. Let $(\mu_i)_{i=0}^\infty$ be as in 2.2 and $\varepsilon > 0$ be such that $\mu(E) < \varepsilon$ implies that $|\xi|(E) < 1$ for each $\xi \in G$. Let λ, M and $D = \{E_i | i = 1,2, \ldots, K\}$ be as in 3.1 with the stipulation (2.1) that for each $i \leqslant M$ we have $\mu_i(E_i) = \mu_i(S)$.

(We will therefore have $M \leq K$.) Let $\delta = \sum_{i=1}^{M} \mu_i$ $(= \mu - \lambda)$ and define $G_1 = \{P_\delta(\xi)|\xi \in G\}$ and $G_2 = \{P_\lambda(\xi)|\xi \in G\}$.

(1) Let $\xi \in G$. Then since $\mu = \delta + \lambda$ and $\delta \wedge \lambda = 0$ we have (by 2.4)

$$\xi = P_\mu(\xi) = P_{\delta+\lambda}(\xi) = P_\delta(\xi) + P_\lambda(\xi) \in G_1 + G_2.$$

(2) If $\eta \in G_2$, then $\eta = P_\lambda(\xi)$ where $\xi \in G$, hence $|\eta| = |P_\lambda(\xi)| \leq |\xi|$ so that G_2 (and similarly G_1) is uniformly absolutely continuous with respect to μ since G is.

(3) If $\eta_i \in G_i$ $(i = 1,2)$, then (by 2.4)

$$\begin{aligned} |\eta_1| \wedge |\eta_2| &= P_\delta(|\eta_1|) \wedge P_\lambda(|\eta_2|) \\ &\leq P_\delta(|\eta_1| + |\eta_2|) \wedge P_\lambda(|\eta_1| + |\eta_2|) = 0. \end{aligned}$$

(4) This is clear since $P_\delta(\xi) \in A_\delta$.

(5) $M + K$ is a bound for G_2.

Let $\eta \in G_2$ where $\eta = P_\lambda(\xi)$ with $\xi \in G$ and $i \in \{1,2,\ldots,K\}$. If $i > M$, then for each $n \leq M$ we have $\mu_n(E_i) = 0$; hence $\mu(E_i) = \lambda(E_i) < \varepsilon$ so that

$$|\eta|(E_i) = |P_\lambda(\xi)|(E_i) \leq |\xi|(E_i) < 1.$$

Now if $i \leq M$, then

$$\mu(E_i) = \lambda(E_i) + \sum_{j=1}^{M} \mu_j(E_i) = \lambda(E_i) + \mu_i(E_i)$$

and since $\mu_i \wedge \lambda = 0$ we have $\mu_i \wedge |\eta| = 0$. Therefore there exists a V_i such that $\mu_i(V_i) = \mu_i(S)$ and $|\eta|(V_i) < 1$. Since both conditions hold if we replace the set V_i by its intersection with E_i we may select $V_i \subseteq E_i$. Now $\mu_i(E_i \sim V_i) = 0$ so that

$$\mu(E_i \sim V_i) = \lambda(E_i \sim V_i) \leq \lambda(E_i) < \varepsilon$$

hence $|\eta|(E_i \sim V_i) \leq |\xi|(E_i \sim V_i) < 1$. Consequently we have

$$\begin{aligned} |\eta|(S) &= \sum_{i=1}^{K} |\eta|(E_i) \\ &= \sum_{i=1}^{M} |\eta|(V_i) + \sum_{i=1}^{M} |\eta|(E_i \sim V_i) + \sum_{i=M+1}^{K} |\eta|(E_i) \\ &< \sum_{i=1}^{M} 1 + \sum_{i=1}^{M} 1 + \sum_{i=M+1}^{K} 1 = M + K. \end{aligned}$$

Note that if $\mu \in \mathrm{ca}(\Sigma)^+$ and $G \subseteq \mathrm{ca}(\Sigma)$, then the projection description of G_1 can be simplified. That is $G_1 = \{\xi_V|\xi \in G\}$, the set of contractions of elements of G to the set V of 3.2 (or any element of Σ which separates δ and λ). Similarly $G_2 = \{\xi_{S\sim V}|\xi \in G\}$.

3.4. Corollary. *If $G \subseteq \mathrm{ba}(F)$ is uniformly absolutely continuous and $\{P_t(\xi)(S)|\xi \in G\}$ is bounded for each $t \in T$, then G is bounded.*

PROOF. For each $i \leqslant M$ let $t_i = \mu_i / \mu_i(S)$ and B_i be a bound for $\{P_{t_i}(\xi)(S) | \xi \in G\}$. Then $B = \sum_{i=1}^{M} B_i$ is a bound for G_1 and therefore $M + K + B$ is a bound for G.

In $\mathrm{ca}(\Sigma)$, the boundedness condition in the hypothesis of 3.4 may be replaced by: $\{\xi(E) | \xi \in G\}$ is bounded for each μ-atom, E. Another consequence of 3.4 is:

3.5. COROLLARY. *A uniformly absolutely continuous set of nonatomic measures is bounded.*

Finally, we obtain the following theorem which can be found in **[4]**.

3.6. COROLLARY. *If $G \subseteq \mathrm{ba}(F)$ is uniformly absolutely continuous and $\{\xi(E) | \xi \in G\}$ is bounded for each $E \in F$, then G is bounded.*

PROOF. By the hypothesis and the boundedness of G_2 it follows that $\{\eta(E) | \eta \in G_1\}$ is bounded for each $E \in F$ and consequently G_1 is bounded since it is contained in the finite dimensional subspace A_δ. Therefore G is contained in the sum of two bounded sets and is bounded.

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