# NEW EXPLICIT FORMULAS FOR THE COEFFICIENTS OF $p$-SYMMETRIC FUNCTIONS 

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#### Abstract

In comparison with Goodman's well-known method and results, we use an alternate method by which we obtain more tractable formulas for the coefficients of $p$-symmetric univalent functions of classes $S^{p}$ and $\Sigma^{p}$ in terms of the coefficients of their associated functions in the class $S$. Our method and results are simple and convenient and they make it easier to compute the coefficients for functions in the classes $S^{p}$ and $\Sigma^{p}$, because we obtain a new recursion formula for them.


1. Introduction. Let $S$ be the class of functions

$$
\begin{equation*}
f(z)=z+a_{2} z^{2}+\ldots \tag{1}
\end{equation*}
$$

that are regular and univalent in the disk $\Delta:|z|<1$, and let $\Sigma$ be the class of functions

$$
\begin{equation*}
F(z)=z+\alpha_{0}+\alpha_{1} / z+\ldots \tag{2}
\end{equation*}
$$

that are meromorphic and univalent for $|z|>1$. According to the "area theorem", the relation

$$
\begin{equation*}
\sum_{n=1}^{\infty} n\left|\alpha_{n}\right|^{2} \leqslant 1 \tag{3}
\end{equation*}
$$

holds. Further, let $S^{p} \subset S, p=1,2, \ldots$, be the class of $p$-symmetric univalent functions

$$
\begin{equation*}
f_{p}(z)=\left[f\left(z^{p}\right)\right]^{1 / p}=z+a_{p+1}^{(p)} z^{p+1}+a_{2 p+1}^{(p)} z^{2 p+1}+\ldots \tag{4}
\end{equation*}
$$

where $f \in S$, and let $\Sigma^{p} \subset \Sigma, p=1,2, \ldots$, be the class $p$-symmetric univalent functions

$$
\begin{equation*}
F_{p}(z)=\frac{1}{f_{p}(1 / z)}=z+\frac{\alpha_{p-1}^{(p)}}{z^{p-1}}+\frac{\alpha_{2 p-1}^{(p)}}{z^{2 p-1}}+\ldots \tag{5}
\end{equation*}
$$

where $f_{p} \in S^{p}$. Again, the area theorem yields the inequality

$$
\begin{equation*}
\sum_{n=1}^{\infty}(n p-1)\left|\alpha_{n p-1}^{(p)}\right|^{2} \leqslant 1 . \tag{6}
\end{equation*}
$$

Explicit expressions for the coefficients $a_{n p+1}^{(p)}$ and $\alpha_{n p-1}^{(p)}$ in terms of the $\left\{a_{n}\right\}$ are desirable and useful. Such formulas have been obtained recently by Goodman [1] who used induction. In this paper we find more tractable
explicit formulas for the coefficients $a_{n p+1}^{(p)}$ and $\alpha_{n p-1}^{(p)}$ in terms of $\left\{a_{n}\right\}$. Our method is based on a classic formula due to Faà di Bruno (cf. [2]-[6]).

In §2 we first give the precise form of the Faà di Bruno formula we need, and then we adapt it for application to coefficient problems that arise from certain operations with power series (superpositions, inversions, etc.). In §3 we find the coefficients $a_{n p+1}^{(p)}$ and $\alpha_{n p-1}^{(p)}$ in terms of the $\left\{a_{n}\right\}$.
2. On a form of the Faà di Bruno formula. The Faà di Bruno formula [2]-[6] gives an explicit form of the $n$th derivative of the composite functions $g(z) \equiv \phi(t) \circ f(z) \equiv \phi[f(z)]$, where the functions $\phi(t), f(z)$ and $g(z)$ are regular in their respective domains. If certain derivatives are denoted by

$$
\begin{equation*}
g_{n} \equiv g^{(n)}(z), \quad \phi_{n} \equiv \phi^{(n)}(t) \quad \text { and } \quad f_{n} \equiv f^{(n)}(z), \quad n=1,2, \ldots, \tag{7}
\end{equation*}
$$

then the classic formula due to Faà di Bruno is

$$
\begin{equation*}
g_{n}=\sum_{k=1}^{n} \phi_{k} B_{n k}\left(f_{1}, f_{2}, \ldots, f_{n-k+1}\right), \tag{8}
\end{equation*}
$$

where

$$
\begin{equation*}
B_{n k}\left(f_{1}, f_{2}, \ldots, f_{n-k+1}\right) \equiv \sum \frac{n!}{\nu_{1}!\nu_{2}!\cdots \nu_{n}!}\left(\frac{f_{1}}{1!}\right)^{\nu_{1}}\left(\frac{f_{2}}{2!}\right)^{\nu_{2}} \cdots\left(\frac{f_{n}}{n!}\right)^{\nu_{n}}, \tag{9}
\end{equation*}
$$

and the sum is taken over all nonnegative integers $\nu_{1}, \nu_{2}, \ldots, \nu_{n}$ for which

$$
\begin{equation*}
\nu_{1}+\nu_{2}+\cdots+\nu_{n}=k, \quad \nu_{1}+2 \nu_{2}+\cdots+n \nu_{n}=n . \tag{10}
\end{equation*}
$$

The functions $B_{n k}\left(f_{1}, \ldots, f_{n-k+1}\right)$, defined in (9)-(10), are homogeneous isobaric polynomials of degree $k$ and of weight $n$ in the variables $f_{1}, f_{2}, \ldots, f_{n-k+1}$ and they are called partial Bell polynomials; the $g_{n}$ in (8) are called complete Bell polynomials in the variables $\phi_{k}, f_{k}, 1 \leqslant k \leqslant n$ ([4]-[6]).

We note that the partial Bell polynomials $B_{n k}\left(f_{1}, \ldots, f_{n-k+1}\right)$ depend only on the variables $f_{s}, 1 \leqslant s \leqslant n-k+1$, while on the right-hand side of (9) we find all the $f_{s}, 1 \leqslant s \leqslant n$. This is a result of the appearance of all $\nu_{s}, 1 \leqslant s \leqslant n$ in (10). Hence for $2 \leqslant k \leqslant n, n \geqslant 2$ in (10) we must have $\nu_{s}=0$ for all $[s \mid n \geqslant s>n-k+1]$. A proof of this fact is contained in the proof we give in establishing the following precise form of the Faà di Bruno formula.

Theorem 1. Let functions $w=\phi(t)$ and $t=f(z)$ be regular in domains $G_{t}$ and $G_{z}$, respectively, where $G_{t}=f\left(G_{z}\right)$. Then the classic Faà di Bruno formula for the nth derivative, $n \geqslant 1$, of the composite function $g(z)=\phi(t) \circ f(z) \equiv$ $\phi(f(z))$ has the (more precise) form

$$
\begin{equation*}
g_{n}=\sum_{k=1}^{n} \phi_{k} B_{n k}\left(f_{1}, \ldots, f_{n-k+1}\right), \tag{11}
\end{equation*}
$$

where the partial polynomials now have the form

$$
\begin{align*}
B_{n k}\left(f_{1}, \ldots,\right. & \left.f_{n-k+1}\right) \\
& \equiv \sum \frac{n!}{\nu_{1}!\cdots \nu_{n-k+1}!}\left(\frac{f_{1}}{1!}\right)^{\nu_{1}} \cdots\left(\frac{f_{n-k+1}}{(n-k+1)!}\right)^{\nu_{n-k+1}} \tag{12}
\end{align*}
$$

and the sum is taken over all nonnegative integers $\nu_{1}, \nu_{2}, \ldots, \nu_{n-k+1}$ satisfying

$$
\begin{gather*}
\nu_{1}+\nu_{2}+\cdots+\nu_{n-k+1}=k \\
\nu_{1}+\nu_{2}+\cdots+(n-k+1) \nu_{n-k+1}=n \tag{13}
\end{gather*}
$$

Proof. For $k=1$ the statement is true because (12) and (13) coincide with (9) and (10), respectively. Let $2 \leqslant k \leqslant n, n \geqslant 2$. By substitution in (10), we obtain

$$
\begin{equation*}
\sum_{s=2}^{n}(s-1) \nu_{s}=n-k \tag{14}
\end{equation*}
$$

But for nonnegative integers $\nu_{s}, 2 \leqslant s \leqslant n$ and $s-1>n-k$, this equation is absurd, unless $\nu_{s}=0$ for all $s>n-k+1$, in order that (10) should be a compatible pair of equations. Hence equations (9) and (10) take the form (12) and (13), respectively. This completes the proof of Theorem 1.

With the more precise formula of Faà di Bruno, (11)-(13), we can now modify the known results that were obtained by using the relations (8)-(10) (see [4]-[6]) to obtain formulas that suit our purpose better.

The Bell polynomials $B_{n k}=B_{n k}\left(f_{1}, \ldots, f_{n-k+1}\right)$ satisfy the recursion formula ([6, p. 136, formula [3k]])

$$
\begin{align*}
& k B_{n k}=\sum_{\mu=k}^{n}\binom{n}{\mu-1} f_{n-\mu+1} B_{\mu-1, k-1}, \\
&  \tag{15}\\
& \qquad 1 \leqslant k \leqslant n, \quad n \geqslant 1, \quad B_{n 0}=0, \quad B_{00}=1 .
\end{align*}
$$

If in (12) we introduce the Taylor coefficients

$$
\begin{equation*}
c_{n}=f_{n} / n!, \quad n=1,2, \ldots \tag{16}
\end{equation*}
$$

and if instead of the polynomials (12) we use the polynomials

$$
\begin{align*}
C_{n k}\left(c_{1}, \ldots,\right. & \left.c_{n-k+1}\right) \\
& =(1 / n!) B_{n k}\left(1!c_{1}, \ldots,(n-k+1)!c_{n-k+1}\right), \quad n \geq 1 \tag{17}
\end{align*}
$$

then (15) acquires the following reduced form:

$$
\begin{align*}
C_{n k} & =\frac{1}{k} \sum_{\mu=1}^{n-k+1} c_{\mu} C_{n-\mu, k-1}, \quad 1 \leqslant k \leqslant n, \quad n \geqslant 1, \quad C_{n 0}=0, \quad C_{00}=1 \\
C_{n k} & =C_{n k}\left(c_{1}, \ldots, c_{n-k+1}\right) . \tag{18}
\end{align*}
$$

Here, of course, we have used (16) and (17). Now (12), (17), and (18) permit us to write the homogeneous isobaric polynomials $C_{n k}\left(c_{1}, \ldots, c_{n-k+1}\right)$ of degree $k$ and weight $n$ in the form

$$
\begin{equation*}
C_{n k}\left(c_{1}, \ldots, c_{n-k+1}\right)=\sum \frac{\left(c_{1}\right)^{\nu_{1}} \cdots\left(c_{n-k+1}\right)^{\nu_{n-k+1}}}{\nu_{1}!\cdots \nu_{n-k+1}!} \tag{19}
\end{equation*}
$$

where the summation runs over all the nonnegative integers $\nu_{1}, \ldots, \nu_{n-k+1}$ satisfying (13). In particular, we note that the last polynomials have the form

$$
\begin{equation*}
C_{n 1}=c_{n}, \quad C_{n n}=(1 / n!) c_{1}^{n}, \quad n \geqslant 1 \tag{20}
\end{equation*}
$$

Now we can give our solution to the problem posed in $\S 1$.
3. An explicit form of the coefficients $a_{n p+1}^{(p)}$ and $\alpha_{n p-1}^{(p)}$ in the coefficients $\left\{a_{n}\right\}$. For an arbitrary $x$, let $(x)_{k}$ denote the factorial polynomial

$$
\begin{equation*}
(x)_{k} \equiv x(x-1) \cdots(x-k+1), \quad k=1,2, \ldots, \quad(x)_{0}=1 \tag{21}
\end{equation*}
$$

a symbol we shall use in our proof of the following new result.
Theorem 2. The coefficients in the expansions (4) and (5) have the following explicit form in terms of the $\left\{a_{n}\right\}$ in the expansion (1):

$$
\begin{array}{ll}
a_{n p+1}^{(p)}=\sum_{k=1}^{n}\left(\frac{1}{p}\right)_{k} C_{n k}\left(a_{2}, \ldots, a_{n-k+2}\right), & n, p=1,2, \ldots, \\
\alpha_{n p-1}^{(p)}=\sum_{k=1}^{n}\left(-\frac{1}{p}\right)_{k} C_{n k}\left(a_{2}, \ldots, a_{n-k+2}\right), & n, p=1,2, \ldots, \tag{23}
\end{array}
$$

where

$$
\begin{equation*}
C_{n k}\left(a_{2}, \ldots, a_{n-k+2}\right)=\sum \frac{a_{2}^{\nu_{1}} \cdots a_{n-k+2}^{\nu_{n-k}}}{\nu_{1}!\cdots \nu_{n-k+1}!} \tag{24}
\end{equation*}
$$

and the sum is taken over all solutions in nonnegative integers $\nu_{1}, \ldots, \nu_{n-k+1}$ of the system (13).

Proof. The function $\phi(t)=t^{m}, \phi(1)=1$, is regular at the point $t=1$ for any arbitrary complex number $m$, and the function $t=\psi(z)=f(z) / z$, where $f(z) \in S$, is regular at $z=0$ where it has the value $t=\psi(0)=1$. Hence, the composite function

$$
g(z)=t^{m} \circ\left(\frac{f(z)}{z}\right) \equiv\left(\frac{f(z)}{z}\right)^{m}, \quad g(0)=1
$$

is regular at $z=0$, and, indeed, in the disk $|z|<1$ since $\psi(z) \neq 0$ in $|z|<1$. Thus we have the following expansion, valid for $f \in S$ and $|z|<1$ :

$$
\begin{equation*}
\left(\frac{f(z)}{z}\right)^{m}=1+\sum_{n=1}^{\infty} g_{n}(m) z^{n}, \quad f(z) \in S \tag{25}
\end{equation*}
$$

The Taylor coefficients $g_{n}(m)=g^{(n)}(0) / n!$ can be found immediately with the help of the Faà di Bruno precise formula (11)-(13)

$$
\phi^{(k)}(1)=(m)_{k}, \quad k \geqslant 1, \quad \text { and } \quad \psi^{(n)}(0)=n!a_{n+1}, \quad n \geqslant 1
$$

If we use these explicit values in (11)-(13), and (17)-(19), we obtain the explicit formula

$$
\begin{align*}
g_{n}(m) & =\frac{1}{n!} \sum_{k=1}^{n}(m)_{k} B_{n k}\left(1!a_{2}, \ldots,(n-k+1)!a_{n-k+2}\right) \\
& =\sum_{k=1}^{n}(m)_{k} C_{n k}\left(a_{2}, \ldots, a_{n-k+2}\right), \quad n=1,2, \ldots, \tag{26}
\end{align*}
$$

where the polynomials $C_{n k}\left(a_{2}, \ldots, a_{n-k+2}\right)$ are given by (24) with the $\nu_{k}$ satisfying (13).
In particular, if in (25) we set $m=1 / p, p=1,2, \ldots$, and replace $z$ by $z^{p}$, $|z|<1$, we obtain the expansion (4). If we write $a_{n p+1}^{(p)}=g_{n}(1 / p)$, then we obtain (22) from (26). If in (25) we set $m=-(1 / p), p=1,2, \ldots$, and replace $z$ by $1 / z^{p},|z|>1$, we obtain the expansion (5). If we write $\alpha_{n p-1}^{(p)}=$ $g_{n}[-(1 / p)]$, then we obtain the formula (23) from (26). This completes the proof of the Theorem 2.

Remark. Theorem 2 holds even when the function (1) is only regular in the disk $|z|<1$. Then the expansions (4) and (5) are convergent, respectively, in the disk $|z|<r^{1 / p}$ and in the ring $r^{-(1 / p)}<|z|<\infty$, where $r$ is the distance from point $z=0$ to the nearest zero of the function $(f(z) / z)$ in $|z|<1$. If $(f(z) / z) \neq 0$ in $|z|<1$, then the expansions (4) and (5) are valid in $|z|<1$ and $|z|>1$, respectively.
If we substitute $c_{\mu}$ for $a_{\mu+1}$ in (18), we find that the polynomials (24) satisfy the recursion formula

$$
\begin{align*}
& C_{n k}=\frac{1}{k} \sum_{\mu=1}^{n-k+1} a_{\mu+1} C_{n-\mu, k-1}, \\
& \qquad \quad 1 \leqslant k \leqslant n, \quad n \geqslant 1, \quad C_{n 0}=0, \quad C_{00}=1,  \tag{27}\\
& C_{n k} \equiv C_{n k}\left(a_{2}, \ldots, a_{n-k+2}\right),
\end{align*}
$$

where (the first and the last polynomials are)

$$
\begin{equation*}
C_{n 1}=a_{n+1}, \quad C_{n n}=(1 / n!) a_{2}^{n}, \quad n \geqslant 1 . \tag{28}
\end{equation*}
$$

The recursion relation (27) is very useful for computing the successive $C_{n k}$; this is one of the main contributions to our study of the coefficients $\alpha_{n p-1}^{(p)}$, $a_{n p+1}^{(p)}$.

For example, from (27) and (28) we obtain the following typical examples of the polynomials (24):

$$
\begin{aligned}
& C_{11}=a_{2}, \quad C_{21}=a_{3}, \quad C_{31}=a_{4}, \quad C_{32}=a_{2} a_{3}, \quad C_{41}=a_{5}, \\
& C_{42}=a_{2} a_{4}+\frac{1}{2} a_{3}^{2}, \quad C_{43}=\frac{1}{2} a_{2}^{2} a_{3}, \quad C_{44}=\frac{1}{24} a_{2}^{4}, \\
& C_{51}=a_{6}, \quad C_{52}=a_{2} a_{5}+a_{3} a_{4}, \quad C_{53}=\frac{1}{2} a_{2}^{2} a_{4}+\frac{1}{2} a_{2} a_{3}^{2}, \\
& C_{54}=\frac{1}{6} a_{2}^{3} a_{3}, \quad C_{55}=\frac{1}{120} a_{2}^{5} .
\end{aligned}
$$

From these equations, we obtain for example

$$
\begin{aligned}
\alpha_{S_{p-1}}^{(p)}= & -\frac{1}{p} a_{6}+\frac{(p+1)}{p^{2}}\left(a_{2} a_{5}+a_{3} a_{4}\right)-\frac{(p+1)(2 p+1)}{2 p^{3}}\left(a_{2}^{2} a_{4}+a_{2} a_{3}^{2}\right) \\
& +\frac{(p+1)(2 p+1)(3 p+1)}{6 p^{4}} a_{2}^{3} a_{3} \\
& -\frac{(p+1)(2 p+1)(3 p+1)(4 p+1)}{120 p^{5}} a_{2}^{5} .
\end{aligned}
$$

This coefficient is essentially the same as that obtained by Goodman [1, formula (10), p. 439] except for the oversight " $a_{2}^{3} a_{4}$ " in his formula rather than the correct " $a_{2}^{3} a_{3}$ ".

## References

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