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ON A THEOREM OF P. S. MUHLY

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ABSTRACT. The purpose of this paper is to show that if $\mathfrak{M}_{\mathfrak{A}}$ is the maximal ideal space of the function algebra induced by a strictly ergodic flow, then almost every point in $\mathfrak{M}_{\mathfrak{A}}$ has a unique representing measure which is concentrated on an orbit. This result enables us to extend some theorems of Muhly to a more general setting.

1. Introduction. Let X be a compact Hausdorff space upon which the real line R acts as a locally compact transformation group. The pair (X, R) is called a *flow* and the translate of an x in X by t is written $x + t$. Let $C(X)$ be the space of all complex-valued continuous functions, and let $\mathfrak{A}$ be the algebra of all functions ϕ in $C(X)$ with the property that, for each x in X, the function $\phi(x + t)$ of t is the boundary function of a function which is bounded and analytic in the upper half-plane. Our reference for the basic facts about such algebras is [**2**]. It was shown that if the flow is strictly ergodic, meaning that there exists a unique probability measure which is invariant under the action of R, then $\mathfrak{A}$ is a Dirichlet algebra on X, that is, $\mathfrak{A}$ contains the constant functions and $\mathfrak{A} + \overline{\mathfrak{A}}$ is uniformly dense in $C(X)$ (see [**8**, Theorem II]). We shall always assume that the flow (X, R) is strictly ergodic. In [**9**], Muhly completely determined the structure of the maximal ideal space $\mathfrak{M}_{\mathfrak{A}}$ of $\mathfrak{A}$ when X is separable. In this paper we shall show that his results are correct in the nonseparable cases. We remark that our results were motivated by §1 and §6 in [**9**].

Let $M(X)$ be the space of all complex regular Borel measures on X, and let $\mathfrak{B}_X$ be the σ-algebra of all Borel subsets of X. If λ is a measure in $M(X)$, then the measure algebra defined by λ will be denoted by $(\mathfrak{B}_X, \lambda)$. Similarly, if $\mathfrak{B}_T$ is the σ-algebra of all Borel subsets of the unit circle T, then $(\mathfrak{B}_T, \mu_T)$ will denote the measure algebra defined by normalized Lebesgue measure μ_T on T (cf. [**4**, pp. 42–45]). A measure in $M(X)$ is called *quasi-invariant* in case its null sets are preserved under the translation. A quasi-invariant measure is said to be *ergodic* in case the only invariant subsets in $\mathfrak{B}_X$ are null or have null complements. Recall that if m represents a point in $\mathfrak{M}_{\mathfrak{A}}$ and if m is not a point mass, then m is an ergodic quasi-invariant measure [**8**, Theorem VI]. Also recall that if m represents a point in $\mathfrak{M}_{\mathfrak{A}}$ and if m is neither a point mass nor the unique invariant probability measure, then the Gleason part contain-

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ing the point represented by m is nontrivial [**9**, Corollary 3.3].

2. The primary result. Our result to be established in this section is as follows:

THEOREM 2.1. *Let m be a representing measure for a point in $\mathfrak{M}_{\mathfrak{A}}$, and assume that m is neither the unique invariant probability measure on X nor a point mass on X. Then m is concentrated on an orbit.*

In order to prove Theorem 2.1, we provide some lemmas. We shall denote the Lebesgue and Hardy spaces based on m by $L^p(m)$ and $H^p(m)$, respectively. The usual Lebesgue and Hardy spaces on the unit circle T will be denoted by $L^p(T)$ and $H^p(T)$, respectively.

LEMMA 2.2. *Let m be a measure as in Theorem 2.1. Then there exists a isometric operator W from $L^2(m)$ onto $L^2(T)$ which has the following properties:*

(i) $W(H^2(m)) = H^2(T)$, $W(L^\infty(m)) = L^\infty(T)$, *and* $W(H^\infty(m)) = H^\infty(T)$;

(ii) *the operator W is induced by an isomorphism τ from $(\mathfrak{B}_X, m)$ onto $(\mathfrak{B}_T, \mu_T)$.*

PROOF. We know that $\mathfrak{A}$ is a Dirichlet algebra on X, and that the Gleason part containing the point represented by m is nontrivial [**8**], [**9**]. It follows from Theorem VI and Corollary 3.1 of [**8**] that $H^\infty(m)$ is a maximal weak-* closed subalgebra in $L^\infty(m)$. Therefore, from the proof of Theorem 1 in [**7**], we can find an isometric operator W from $L^2(m)$ onto $L^2(T)$ which has property (i). We also see that W is multiplicative on $L^\infty(m)$. Hence it follows from the multiplication theorem (cf. [**4**, p. 45]) that W is induced by an isomorphism τ from $(\mathfrak{B}_X, m)$ onto $(\mathfrak{B}_T, \mu_T)$.

Although the following lemma was essentially proved in [**9**, Theorem I], we provide here an elementary proof.

LEMMA 2.3. *Let $\{\alpha_t\}_{t\in R}$ be a continuous one-parameter group of conformal mappings of the open unit disc onto itself. Then $\alpha_t(z)$ may be considered as a continuous function from $R \times T$ to T which defines a flow on T.*

PROOF. For any s in R, let t_n be an arbitrary sequence converging to s. Since $t \to \alpha_{-t}(0)$ is continuous, there is an $r < 1$ such that $|\alpha_{-t_n}(0)| < r$ for all n. Then, since $\alpha_t(z)$ is a fractional linear mapping, we can choose an $r' > 1$ such that $\{\alpha_{t_n}(z)\}$ are analytic and uniformly bounded on $\{|z| < r'\}$. So the functions $\{\alpha_{t_n}(z)\}$ form a normal class. Hence there exists a subsequence which converges uniformly to $\alpha_s(z)$ on the closed unit disc $\{|z| \leq 1\}$. So $(t, z) \to \alpha_t(z)$ is a continuous function from $R \times T$ to T. From our hypothesis, it can also be seen that $\{\alpha_t\}_{t\in R}$ defines a flow on T.

LEMMA 2.4. *Let $\{\alpha_t\}_{t\in R}$ be as in Lemma 2.3. Assume that μ_T is quasi-invariant under the action of α_t, and assume that there is a t in R such that α_t is not the identity transformation. Then we can choose an $a > 0$, an open subset E of T, and a closed interval $I = [0, s]$ such that, for any compact subset G of E*

with $\mu_T(G) > 0$,

$$\mu_T\Big(\bigcup \{\alpha_t(G); t \in I\}\Big) \geqslant \mu_T\Big(\bigcup \{\alpha_{t_n}(G); n = 1,2, \dots\}\Big)$$
$$> a \tag{1}$$

where $\{t_n\}$ *is a countable dense subset of* I.

PROOF. By our assumption, we can find an $s > 0$ and a point z_0 in T such that $\alpha_t(z_0) \neq z_0$ for each t in $(0, s]$. Since $t \to \alpha_t(z_0)$ is continuous, it can be easily seen that there exist an $a > 0$ and an open neighbourhood E of z_0 such that $E \cap \alpha_s(E) = \varnothing$ and

$$\mu_T(\{\alpha_t(z); t \in I\} \setminus (E \cup \alpha_s(E))) > a \tag{2}$$

for any z in E. Let G be a compact subset of E with $\mu_T(G) > 0$. Then we put

$$\begin{aligned} H_1 &= \bigcup \{\alpha_t(G); t \in I\}, \\ H_2 &= \bigcup \{\alpha_{t_n}(G); n = 1,2, \dots\}, \\ F_1 &= H_1 \setminus H_2, \\ F &= F_1 \setminus (E \cup \alpha_s(E)). \end{aligned} \tag{3}$$

Notice that H_1, H_2, F_1, and F are Borel subsets. We claim that $\mu_T(F) = 0$. Suppose not. Then, since μ_T is quasi-invariant, we have

$$\mu_T(\alpha_{-t}(F)) = \int_T \chi_F(\alpha_t(z))\, d\mu_T(z) > 0$$

for each t in R, where χ_F denotes the characteristic function of F. This implies that

$$\int_0^s \int_T \chi_F(\alpha_t(z))\, d\mu_T(z)\, dt > 0.$$

So it follows from Fubini's theorem that there exists a point z_1 in T such that $\{t; t \in I, \alpha_t(z_1) \in F\}$ has positive linear measure. Let z be an arbitrary point in G. Since F is a subset of $\{\alpha_t(z); t \in I\}$, it is easy to see that $\{t; t \in I, \alpha_t(z) \in F\}$ has positive linear measure. Therefore, from the fact $\mu_T(G) > 0$, we have

$$\int_T \int_0^s \chi_F(\alpha_t(z)) \chi_G(z)\, d\mu_T(z)\, dt > 0.$$

Then there is a t in I such that $\mu_T(\alpha_{-t}(F) \cap G) > 0$, so we have $\mu_T(\alpha_t(G) \cap F_1) > 0$. On the other hand, since μ_T is quasi-invariant, it follows from the proof in [**2**, Theorem 4] that there is an n such that $\mu_T(\alpha_{t_n}(G) \cap F_1) > 0$. Thus we have a contradiction. Therefore, from (2) and (3), we obtain the desired inequality (1).

For a subset N of X and a subset J of R, we write

$$N + J = \{x + t; (x, t) \in N \times J\}.$$

We notice that if both N and J are compact, then $N + J$ is also compact, so is a Borel subset.

LEMMA 2.5. *Let m be an ergodic quasi-invariant positive measure in $M(X)$, and let I be a compact subset of R. Suppose that there exists an $a > 0$ and a subset M in $\mathfrak{B}_X$ with $m(M) > 0$ such that $m(N + I) > a$ for any compact subset N of M with $m(N) > 0$. Then m is concentrated on an orbit.*

PROOF. We claim that there exists a point x_0 in M such that $m(x_0 + I) \geqslant a$. Suppose not; let x be an arbitrary point in M. Then, since m is regular, we can find an open set U such that $x + I \subset U$ and $m(U) < a$. Since I is compact in R, it can be easily seen that there is a compact neighbourhood $V(x)$ of x with $V(x) + I \subset U$. Notice that we may assume that M is compact. Since $\{V(x); x \in M\}$ is a covering of M, we can choose a compact neighbourhood $V(x)$ such that $m(M \cap V(x)) > 0$ and $m((M \cap V(x)) + I) < a$. This contradicts our hypothesis. Therefore it follows from the definition of ergodicity that m is concentrated on the orbit of some point x_0 in M.

PROOF OF THEOREM 2.1. We use the modification argument of the proof in [**9**, Theorem I]. Let W be an isometric operator from $L^2(m)$ onto $L^2(T)$ as in Lemma 2.2, and let τ be the isomorphism from $(\mathfrak{B}_X, m)$ onto $(\mathfrak{B}_T, \mu_T)$ associated with W. We set, for any f in $H^\infty(T)$, $\tilde{T}_t(f) = WT_tW^{-1}(f)$ where $T_t\phi$ denotes the function whose value at x is $\phi(x + t)$. Then we see easily that $\{\tilde{T}_t\}$ is a weak-* continuous automorphism from $H^\infty(T)$ onto itself. Therefore it follows from [**1**, Theorem 1] that there exists a continuous one-parameter family of conformal mappings $\{\alpha_t\}_{t\in R}$ on the open unit disc onto itself such that

$$\tilde{T}_t(f)(z) = f(\alpha_t(z)) \tag{4}$$

for any f in $H^\infty(T)$. By Lemma 2.3, $\{\alpha_t\}_{t\in R}$ defines a flow on T. Notice that we may assume that there is a t in R such that α_t is not the identity transformation. Since $H^\infty(T)$ is a weak-* Dirichlet algebra, (4) holds for any f in $L^\infty(T)$. By (4) and the properties of the isomorphism τ, it can be easily seen that

$$\begin{aligned} m(\tau^{-1}(F) + t) &= \mu_T\big(\tau(\tau^{-1}(F) + t)\big) \\ &= \mu_T(\alpha_t(F)) \end{aligned} \tag{5}$$

for any F in $\mathfrak{B}_T$ with $\mu_T(F) > 0$, and for any t in R. Since m is quasi-invariant, it follows from (5) that μ_T is quasi-invariant under the action $\{\alpha_t\}_{t\in R}$. Let a, E, and I be as in Lemma 2.4. Then we can show that $m(N + I) > a$ for any compact subset N of $\tau^{-1}(E)$ with $m(N) > 0$. Indeed, let G be a compact subset of $\tau(N)$ with $\mu_T(G) > 0$, and let $\{t_n\}$ be a countable dense subset of I. Then we have

$$\begin{aligned} m(N + I) &\geqslant m(N + \{t_n\}) \\ &\geqslant \mu_T\Big(\bigcup \{\alpha_{t_n}(G); n = 1, 2, \dots\}\Big) > a \end{aligned}$$

by (5) and Lemma 2.4. By [8, Theorem VI], m is an ergodic quasi-invariant measure. Therefore it follows from Lemma 2.5 that m is concentrated on an orbit, so the proof is complete.

3. Applications. In this section we collect some theorems following from our Theorem 2.1. They are at most minor variations of Muhly's results, so the proofs will be omitted.

From the proof of Step II in [9, Theorem II] and Theorem 2.1 we can see the following:

THEOREM 3.1. *Let m be a representing measure for a point in $\mathfrak{M}_{\mathfrak{A}}$, and assume that m is neither the unique invariant probability measure on X nor a point mass on X. Then there is a unique x in X and a unique $y > 0$ such that $m = \delta_x * P_{iy}$ where $\delta_x * P_{iy}$ is defined to be the measure such that*

$$\int f(w)\, d(\delta_x * P_{iy}(w)) = \int_{-\infty}^{\infty} f(x+t) \frac{y}{\pi(y^2+t^2)}\, dt$$

for all f in $C(X)$.

Next we let $\mathbf{D}$ denote the quotient space obtained from $X \times [0, 1]$ by identifying the slice $X \times \{0\}$ to a point, and for a point x_0 in X, let $\mathbf{H}_{x_0}$ denote the quotient space obtained from $X \times [0, 1]$ by identifying the closed set $(X \times \{0\}) \cup (\{x_0\} \times [0, 1])$ to a point. Then, by the same way as in the proof of Theorem II of [9] we can present the characterization of the maximal ideal space of $\mathfrak{A}$ in the case where X is nonseparable.

THEOREM 3.2. *Let σ be the unique invariant probability measure on X. Then the following properties hold:*

(i) *If σ is not a point mass on X, then the maximal ideal space $\mathfrak{M}_{\mathfrak{A}}$ is homeomorphic to the space $\mathbf{D}$ defined above.*

(ii) *If σ is a point mas δ_{x_0} on X, then the maximal ideal space $\mathfrak{M}_{\mathfrak{A}}$ is homeomorphic to the space $\mathbf{H}_{x_0}$ defined above.*

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